ON EXTENDING GROUP ACTIONS FROM SURFACES TO THREE SPHERE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

DAMLA ERDOĞAN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

JUNE 2022

Approval of the thesis:

ON EXTENDING GROUP ACTIONS FROM SURFACES TO THREE SPHERE

submitted by DAMLA ERDOĞAN in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Halil Kalıpçılar
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Yıldıray Ozan
Head of Department, **Mathematics** ______________

Assoc. Prof. Dr. Semra Pamuk
Supervisor, **Department of Mathematics, METU** ______________

Examining Committee Members:

Prof. Dr. Ferihe Atalan
Department of Mathematics, METU ______________

Assoc. Prof. Dr. Semra Pamuk
Department of Mathematics, METU ______________

Assoc. Prof. Dr. Mehmet Fırat Arıkan
Department of Mathematics, Atılım Üniversitesi ______________

Date:

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: Damla Erdoğan

Signature :

ABSTRACT

ON EXTENDING GROUP ACTIONS FROM SURFACES TO THREE SPHERE

Erdoğan Damla,
M.S., Department of Mathematics
Supervisor : Assoc. Prof. Dr. Semra Pamuk

June 2022, 66 pages

Even though surfaces are the most elementary objects in geometry and topology, because of the variety of structures they have it is still an active area of research. The symmetries of surfaces have been studied for a long time. One of the compelling questions is which of these symmetries can be extended to handlebodies and 3-sphere. In this thesis, we are focusing on the symmetries of surfaces which can be embedded into the symmetries of 3-sphere. The aim is to give an overview of the problem with some background and present some of the known results with proofs.

Keywords: Finite group actions, extendable map, symmetry of surfaces, symmetry of 3-sphere

ÖZ

YÜZEYLERDEKİ GRUP ETKİLERİNİN ÜÇ BOYUTLU KÜREYE GENİŞLETİLMESİ ÜZERİNE

Erdoğan Damla,
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Semra Pamuk

Haziran 2022 , 66 sayfa

Yüzeyler geometri ve topolojinin en temel nesneleri olmasına rağmen, üzerlerine konabilecek yapıların çeşitliliğinden hala aktif bir araştırma alanıdır. Yüzeylerin simetrileri uzun bir süredir çalışılmaktadır. Zorlayıcı sorulardan biri de bu simetrilerden hangisinin kulplara ve üç boyutlu küreye genişletilebileceğidir. Bu tezde, üç boyutlu küre simetrilerine gömülebilecek yüzey simetrilerine odaklanıyoruz. Amacımız biraz alt yapı ile probleme genel bir özet vermek ve bazı bilinen sonuçları kanıtlarıyla birlikte sunmaktır.

Anahtar Kelimeler: sonlu grup etkileri, genişletilebilir fonksiyon, yüzeylerin simetrisi, 3 boyutlu kürenin simetrisi

To my beloved family...

ACKNOWLEDGEMENTS

First of all, I would like to sincerely thank to my supervisor Assoc. Prof. Semra Pamuk for her valuable guidance and motivation during our studies. I am also very grateful to her for introducing this area of research to me, which I have been deeply interested in. Also, her efforts were so valuable to me during this process, which made this thesis possible.

I am also thankful to my instructors who helped me to figure out my research area of interest before my graduate studies. Their guidance means a lot to me.

My dear friends and my sweet sister also have been so encouraging. They always knew how to cheer me up when I needed it. Their moral support helped me proceed.

Last but not the least, I am grateful to my parents for their unconditional love and support. Their belief in me will hopefully make anything possible.

TABLE OF CONTENTS

ABSTRACT v

ÖZ vi

ACKNOWLEDGMENTS viii

TABLE OF CONTENTS ix

LIST OF FIGURES xi

CHAPTERS

1 INTRODUCTION 1

2 BACKGROUND 3

2.1 A Glimpse on Group Actions 3

2.2 An Overview of Knots and Links 4

2.2.1 Wirtinger Presentation 7

2.3 Heegaard Splittings 9

2.4 Branched Coverings 11

2.4.1 Cyclic Branched Covers of S^3 11

2.5 A Short Review of Surgery Theory 16

2.5.1 Surgery Instructions in S^3 16

2.6 Seifert Manifolds 19

2.7 Orbifolds 20

2.7.1 Orbifold coverings 23

2.7.2 Fundamental group of an orbifold 24

2.7.3 Classification of some two and three dimensional orbifolds 25

2.7.4 Handlebody Orbifolds 26

2.8 Orbifolds in Dunbar's List 28

2.8.1 Seifert Fibered Spherical Orbifolds in Dunbar's List 28

2.8.2 Non-Fibered Spherical Orbifolds in Dunbar's List . 32

3 EXTENDING FINITE GROUP ACTIONS ON SURFACES OVER S^3 35

3.1 Maximum order of groups with extendable actions 36

3.2 Maximum orders of abelian and cyclic groups with extendable actions 42

Case 1: 44

Case 2: 45

4 SOME EXAMPLES OF ACTIONS EXTENDING TO S^3 53

5 CONCLUSIONS 63

REFERENCES 65

LIST OF FIGURES

FIGURES

Figure 2.1 (a) an overpass which is not maximal & (b) a maximal overpass [1] 5

Figure 2.2 (a) trefoil knot & (b) figure-eight knot [1] 5

Figure 2.3 some examples of tangles from [1] 5

Figure 2.4 Example of rational tangles [1] 6

Figure 2.5 torus knots of type $(2, 5)$ and $(5, 6)$ resp. [14] 7

Figure 2.6 on computation of a knot group [18] 8

Figure 2.7 trefoil knot [14] 8

Figure 2.8 computation of the knot group of a trefoil [18] 9

Figure 2.9 handlebody of genus 3 [15] 9

Figure 2.10 handlebody of genus 3 homeomorphic to previous figure 10

Figure 2.11 construction of a cyclic branched cover over a knot -1- [14] 12

Figure 2.12 construction of a cyclic branched cover over a knot -2- [14] 12

Figure 2.13 construction of a cyclic branched cover over a knot -3- [14] 13

Figure 2.14 construction of a cyclic branched cover over a knot -4- [14] 13

Figure 2.15 n-fold cyclic cover for $n = 3$[14] 14

Figure 2.16 construction of an n-fold cyclic branched cover [14] 15

Figure 2.17 3-fold cyclic branched covering of S^3 along the figure-eight knot [14] 15

Figure 2.18 on Lemma 2.5.1[15] 17

Figure 2.19 on Lemma 2.5.1[15] 18

Figure 2.20 Surgery description of a Seifert manifold [15] 20

Figure 2.21 pillowcase [3] 22

Figure 2.22 3-discal orbifolds [21] 27

Figure 2.23 obtaining a handlebody orbifold from quotients of 3-balls and a 1-handle 28

Figure 2.24 **Table 1:** fibered spherical orbifolds fibered over 2-sphere [4],[21] . 30

Figure 2.25 **Table 2:** fibered spherical orbifolds fibered over 2-disc [4],[21] . . 31

Figure 2.26 **Table 3:** nonfibered spherical orbifolds [5],[21] 33

Figure 2.27 fundamental group computation for an orbifold in Table 3 [21] . . . 34

Figure 3.1 singular set of the orbifolds H_i/G [21] 37

Figure 3.2 singular set of the orbifold S^3/G [21] 37

Figure 3.3 nonfibered spherical orbifolds with a singularity type $(2,2,2,3)$ [20] 38

Figure 3.4 spherical orbifolds with a two-bridge link singular set [20] 39

Figure 3.5 the handlebody orbifold $V_g/\mathbb{Z}_{2g}$ [20] 49

Figure 3.6 preimage of X in V_3 [20] 49

Figure 3.7 the arc γ on ∂X [20] 50

Figure 3.8 α_i and β_i on D [20] 50

Figure 3.9 the knots K_1 and K_2 in S^3 [20] 50

Figure 3.10 three different positions of K_0 and D [20] 51

Figure 4.1 $\mathbb{Z}_{2g-2}$ action on V_5[20] 56

Figure 4.2 an oriented pair of pants and an oriented circle 58

Figure 4.3 obtaining M' from the pair of pants 58

Figure 4.4 θ-graph for the link with three components [20] 60

Figure 4.5 θ-graph for trefoil [20] 60

CHAPTER 1

INTRODUCTION

Given a group G and a surface Σ, we say that a group G acts on Σ if there is an injection $G \hookrightarrow Sym(\Sigma)$, where $Sym(\Sigma)$ represents symmetries of the surface, for example we consider it as self-homeomorphisms of the surface. The symmetries of surfaces have been investigated for a long time. If we consider Σ as a Riemmann surface, then we can consider $Sym(\Sigma)$ as group of automorphisms of Σ, which are conformal self-homeomorphims and denoted by $Aut(\Sigma)$. It is also known that for any G action on a compact orientable surface by orientation-preserving homeomorphisms there exists a complex structure on the surface so that the group action is given as automorphisms of surface. Schwarz proved at the end of nineteenth century that $Aut(\Sigma_g)$ of a compact Riemann surface Σ_g of genus $g \geq 2$ is finite, and then Hurwitz [7] showed by using Riemann-Hurwitz formula that its order is at most $84(g-1)$. This bound is attained for infinitely many g, the smallest example is Klein's quartic, with genus 3 that has $84(3-1) = 168$ automorphisms. In general, for a fixed genus g it is hard to determine the maximum order. So we have a bound for the order of the finite groups acting on surfaces. To elaborate one can ask what is the maximum order of all finite cyclic or abelian groups that can act on surface of genus g? It has been shown that the $4g+2$ is the bound for cyclic [17] and $4(g+1)$ is the bound for abelian [8].

Same questions are also asked for handlebodies, what is the maximum order of all finite (cyclic, abelian) groups which can act on handlebody H_g of genus g? It has been shown by Zimmermann[22] that the bound for the order of all finite groups that act on a genus H_g is $12(g-1)$. A handlebody orbifold theory has been developed to show this result and also it has been shown that the bound for cyclic ones is $2g+2$

for g even and $2g$ for g odd.

One of the natural questions to ask is which of the finite group actions on a surface Σ_g of genus g extends to a handlebody H_g . Then we can also ask when these actions can be embedded in S^3? In this thesis, we will focus on the latter question and will try to give an overview of the known results and the techniques used in the proofs. More elaborately, suppose we are given a finite group G acting on a genus g surface Σ_g and an embedding $\sigma : \Sigma_g \to S^3$ such that G acts on (S^3, Σ_g) such that the restriction of the action on Σ_g is the given action of G on Σ_g. In this case the action of G on Σ_g is called *extendable* over S^3 via σ.

Note that the embedding $\sigma : \Sigma_g \to S^3$ can be knotted, but we will focus on the unknotted embeddings, i.e. each component of $S^3 \setminus \Sigma_g$ is a handlebody. So for each extendable action of G we have a G-invariant Heegaard splitting of S^3. Likewise, one can define an action of G on H_g to be extendable, and an embedding $e : H_g \to S^3$ to be unknotted if the complement $S^3 \setminus H_g$ is also a handlebody. For each g, an unknotted embedding is unique up to isotopy of S^3 and automorphisms on Σ_g or H_g.

In the search of such extendable actions one first tries to give the maximum order of a finite (cyclic and abelian) groups acting on Σ_g which extends to S^3 with respect to an unknotted embedding. The methods depend on orbifold theory, because the problem of finding such actions can be translated to finding some 2-orbifolds in certain spherical 3-orbifolds, which will be explained in Chapter 3. In chapter 2, the necessary background on group actions, low dimensional topology and orbifold theory will be given in order to understand the proofs in Chapter 3. In Chapter 4, we will give some examples of extendable actions.

CHAPTER 2

BACKGROUND

In this chapter we will introduce some definitions and theorems as background material which will be needed throughout the thesis.

2.1 A Glimpse on Group Actions

In this section, we give some basic notions of group actions on a manifold. More specifically, we will deal with the finite group actions on mostly 2-manifolds.

Definition 2.1.1. *An **action** of a group G on a manifold M is a continuous map*

$$\theta\colon\ G \times M \to M$$

which satisfies for every $g, h \in G$ and $x \in M$

(i) $\theta(1, x) = x$
(ii) $\theta(gh, x) = \theta(g, \theta(h, x))$

We call an action **effective (or faithful)** if the condition $g \cdot x = x$ for all $x \in M$ implies that g is the identity element. Throughout this thesis all actions are assumed to be effective.

For example, the action of the cyclic group $\mathbb{Z}_n$ on the sphere S^2 acting by $\frac{2\pi}{n}$ - rotations of the sphere with rotation of axis from the north pole to the south pole is an effective group action. The action is actually orientation-preserving since it is by rotations.

Definition 2.1.2. *Given an action of* G *on* M*, the quotient space* M/G *endowed with the quotient topology and the equivalence relation* $x \sim gx$ *is called the* ***orbit space*** *of the action of* G *on* M*. Moreover, the quotient map* $\pi : M \to M/G$ *is called the* ***orbit map****.*

Example 2.1.1. *The antipodal map* $x \mapsto -x$ *on* S^2 *gives an orientation reversing action of* $\mathbb{Z}_2$*. Moreover, the orbit space of this action is the real projective plane* $S^2/\mathbb{Z}_2 \cong \mathbb{R}P^2$*.*

Definition 2.1.3. *For every point* x *of* M*, the subgroup of* G *consisting of the elements which fixes the point* x *under this action is called the* ***isotropy subgroup*** *of* x*, denoted by* G_x*. That is,*

$$G_x = \{g \in G \mid gx = x\}$$

Moreover, if each isotropy subgroup is trivial, then the action of G *is said to be* ***free****.*

Definition 2.1.4. *A group* G *is said to act* ***properly discontinuously*** *on* M *if for every* $x \in M$ *there is a neighborhood* U *of* x *such that* $gU \cap U \neq \emptyset$ *only for finitely many group elements* $g \in G$*.*

The group action given in the Example 2.1.1 is also an example of both a properly discontinuous and free action.

2.2 An Overview of Knots and Links

As we know that a knot is an embedding of S^1 into $\mathbb{R}^3$ and we always represent knots as diagrams in $\mathbb{R}^2$ by means of a regular projection.

Definition 2.2.1. *Let a diagram of a knot* K *be given. An* ***overpass*** *in a diagram is defined to be a subarc of the knot passing over at least one crossing but not passing under a crossing.*

A ***maximal overpass*** *is the overpass which is the longest, i.e. both endpoints of the arc ends before an undercrossing [1].*

Definition 2.2.2. *The* ***bridge number*** *of a knot is the number of maximal overpasses among all projections [1].*

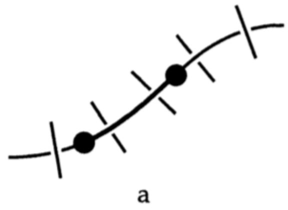


Figure 2.1: (a) an overpass which is not maximal & (b) a maximal overpass [1]

Remark 2.2.1. *If a knot has bridge number* 1*, then it is the unknot.*

Example 2.2.1. *Both the trefoil and the figure-eight knot have bridge number* 2*, where the corresponding bridges are shown in Figure 2.2.*

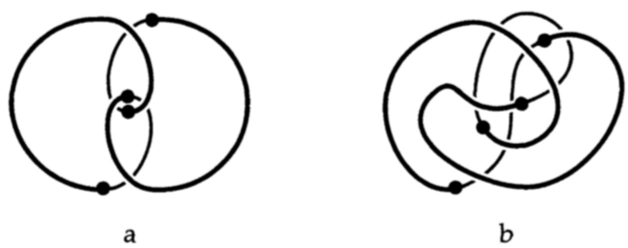


Figure 2.2: (a) trefoil knot & (b) figure-eight knot [1]

Definition 2.2.3. *A part of a knot or link projection intersecting a circular region four times is called a* ***tangle*** *[1].*

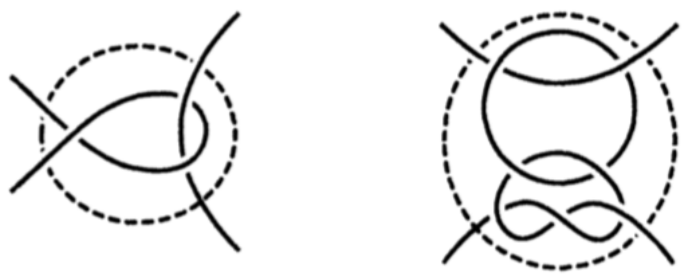


Figure 2.3: some examples of tangles from [1]

Let two horizontal strings be given. Then one can wind the strings around each other a number of times. Let us call this operation as "twist". Also one can rotate the horizontal tangle by $90°$ to get a vertical tangle and then continue twisting the ends of the vertical tangle a number of times again. Let us call this operation as "rotation".

Definition 2.2.4. *(Construction of rational tangles) Let two horizontal strings be given. The tangle obtained by the applying the twist & rotation operations in a sequence is called a* ***rational tangle*** *[1].*

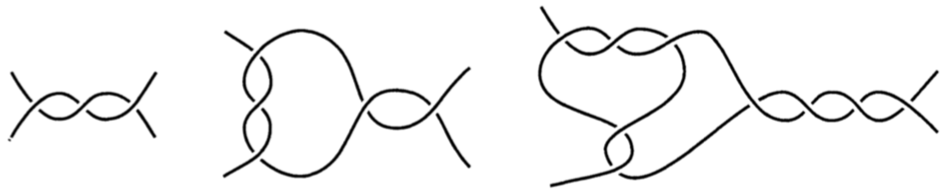


Figure 2.4: Example of rational tangles [1]

In Figure 2.4, one can see the stages of obtaining the rational tangle on the right-hand side. The operations are 3 twists, 1 rotation, 2 twists, 1 rotation, and then 4 twists in the opposite direction. Or simply one can use the notation $3\ 2\ -4$, which counts the twists only.

In short one can represent a rational tangle by the notation $abcd...kl$ where each letter stands for a twist and after each letter the tangle is rotated.

There is a one to one correspondence between rational tangles and the set of natural numbers $\mathbb{Q}$ by the continued fraction

$$l + \cfrac{1}{k + \cfrac{1}{\ddots + \cfrac{1}{d + \cfrac{1}{c + \cfrac{1}{b + \cfrac{1}{a}}}}}}$$

for the tangle $abcd...kl$. Therefore, two rational tangles are the same if and only if their corresponding continued fractions are the same [1].

Definition 2.2.5. *Let p and q be relatively prime. The knot $T_{p,q}$ of type (p, q) is called a* ***torus knot*** *if it wraps the solid torus in the direction of its longitude p-times and in the direction of its meridian q times.*

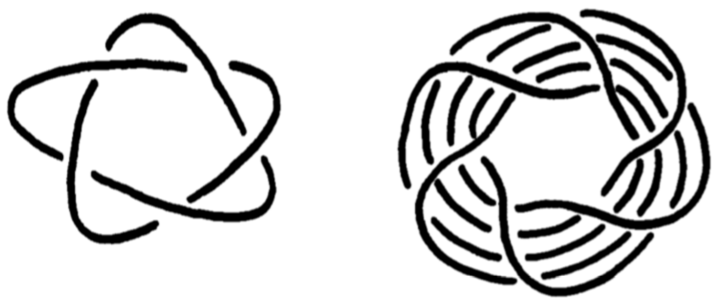


Figure 2.5: torus knots of type $(2, 5)$ and $(5, 6)$ resp. [14]

2.2.1 Wirtinger Presentation

Let a knot (or a link) K in $\mathbb{R}^3$ or (S^3) be given. Then one can associate a group to K, which is the fundamental group of the knot complement $\mathbb{R}^3 \setminus K$. This is called the **knot group** [18].

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the oriented arcs in the diagram of the knot K. (The arcs are being labelled so that its endpoints are undercrossings.) Then $\pi_1(\mathbb{R}^3 \setminus K)$ is generated by the loops x_i which go around these arcs. (Figure 2.6)

- If the crossing type is as in part (1) of the Figure 2.6, then the curve $x_i x_j^{-1} x_{i+1}^{-1} x_j$ contracts to a point. Therefore, $x_i x_j^{-1} x_{i+1}^{-1} x_j = 1$ must hold, i.e.
$$x_j x_i = x_{i+1} x_j.$$
- Similary, if the crossing type is as in part (2) of the Figure 2.6, then the curve $x_i x_j x_{i+1}^{-1} x_j^{-1}$ contracts to a point. Therefore, $x_i x_j x_{i+1}^{-1} x_j^{-1} = 1$ must hold, i.e.
$$x_i x_j = x_j x_{i+1} \text{ [18].}$$

Note that the two other possibilities of the crossing type give the opposite orientation of K and they give the same relations as above.
For n-crossings in the diagram there will be n-arcs. For those n-arcs, there will be n-generators, say $x_1, x_2, ..., x_n$. Also there will be n-relations coming from each crossing, say $r_1, r_2, ..., r_n$.

Hence the knot group is given by $\pi_1(\mathbb{R}^3 \setminus K) =< x_1, x_2, ..., x_n | \, r_1, r_2, ..., r_n >$ [14].

Example 2.2.2. *Let us compute the knot group of the trefoil with the given orientation and generators as in Figure 2.8.*

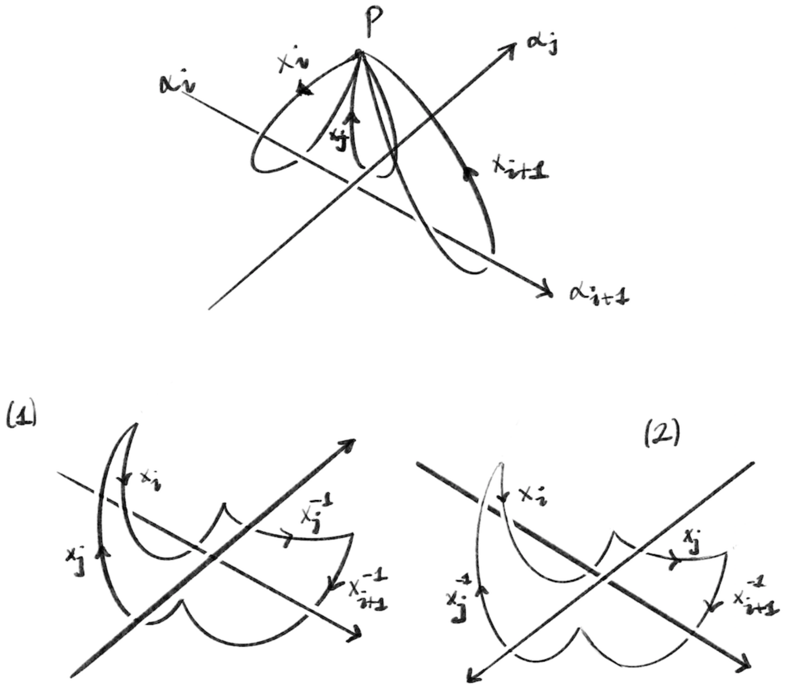


Figure 2.6: on computation of a knot group [18]

Let A, B, C be the crossing points and let $\alpha_1, \alpha_2, \alpha_3$ be the arcs on the knot. The curves drawn around the points A, B, C retracts to a point. Therefore we have the following relations:

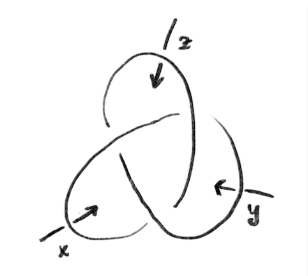


Figure 2.7: trefoil knot [14]

- *For A: $xz^{-1}y^{-1}z = 1$ (1)*
- *For B: $zy^{-1}x^{-1}y = 1$ (2)*
- *For C: $yx^{-1}z^{-1}x = 1$ (3)*

From (1), we have $x = z^{-1}yz$. Substitute this in (2) and (3) to get $zyz = yzy$. We observe that $(yzy)^2 = (zyz)(zyz) = (zy)^3$. Let $a := yzy$ and $b := zy$. Then the

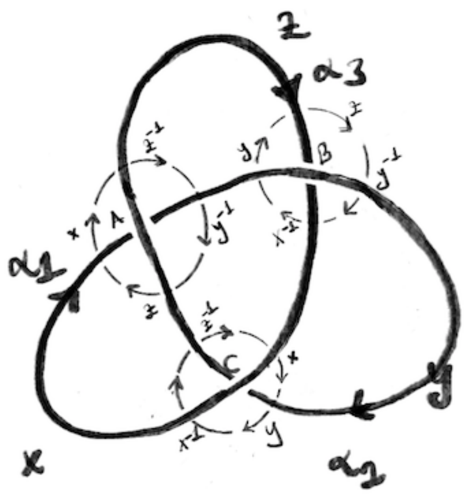


Figure 2.8: computation of the knot group of a trefoil [18]

generators y and z can be obtained from a and b since $y = ab^{-1}$ and $z = b^2a^{-1}$. Similarly, x can be written in terms of a and b. Therefore, the knot group of trefoil is as follows: $\pi_1(\mathbb{R}^3 \setminus K) =< x, y, z| \; xz^{-1}y^{-1}z = 1, \; zy^{-1}x^{-1}y = 1, \; yx^{-1}z^{-1}x = 1 >$ $=< a, b| \; a^2 = b^3 >$.

2.3 Heegaard Splittings

Definition 2.3.1. *The orientable* 3*-manifold with boundary is called a* ***handlebody*** *of genus g if it is obtained by gluing g copies of* 1*-handles of the form $D^2 \times [0,1]$ to a* 3*- ball D^3 by attaching the boundary disks of the handles to the boundary sphere $\partial D^3 = S^2$ [14].*

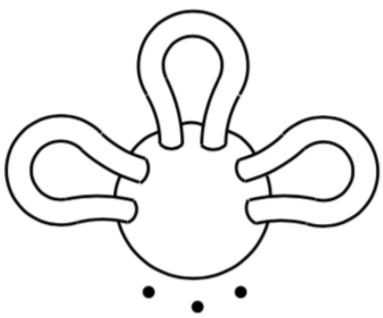


Figure 2.9: handlebody of genus 3 [15]

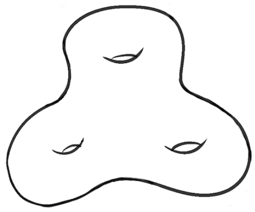


Figure 2.10: handlebody of genus 3 homeomorphic to previous figure

Remark 2.3.1. *Two handlebodies are homeomorphic if and only if they have the same genus g.*

Definition 2.3.2. *[14] Let two handlebodies H and H' of genus g be given. Let $f : \partial H \to \partial H'$ be an orientation-reversing homeomorphism. Consider the closed orientable 3-manifold M obtained by attaching H and H' along their boundaries by the homeomorphism f, i.e. $M = H \cup_f M'$. This handlebody decomposition of the closed orientable manifold M is called a* ***Heegaard splitting*** *of M of genus g.*

Theorem 2.3.1. *Every closed orientable 3-manifold M has a Heegaard splitting.*

Proof. [14] Let T be a triangulation of M. Let M_1 be the one-skeleton of T and $\hat{M}_1$ be the dual one-skeleton of in a barycentric subdivision. This consists of the vertices and the edges which do not meet M_1. Consider the simplicial neighborhoods N_1 and $\hat{N}_1$ of M_1 and $\hat{M}_1$ respectively. Then $N1$ and $\hat{N}_1$ are 3-manifolds with common boundary and $N1 \cup \hat{N}_1 = M$. Only things that needs to be shown is that they are actually handlebodies. Now, since M_1 is a graph therefore it has a maximal tree which contains every vertex. The simplicial neighborhood of that maximal tree is a 3- ball D^3. The simplicial neighborhood of the remaining 1-simplex gives a 1-handle. After attaching the the 1-handle to the 3-ball, one obtains N_1. Therefore, N_1 is a handlebody. Same arguments apply to $\hat{N}_1$ also. Hence, M has a Heegaard splitting.

□

Example 2.3.1. *Consider two solid tori $S^1 \times D^2$. Let the homeomorphism between their boundaries $f : S^1 \times S^1 \to S^1 \times S^1$ be given so that interchanges the copies of S^1. This gives a Heegaard splitting of S^3 since $S^3 = \partial D^4 = \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2) = (S^1 \times D^2) \cup (D^2 \times S^1)$.*

2.4 Branched Coverings

This section is based on Chapter 10 of [14].

Definition 2.4.1. *Let M and N be two compact manifolds of dimension n and let $A \subset M$ and $B \subset N$ be two submanifolds of dimension $n-2$. Then a continuous function $f : M \to N$ is called a* ***branched covering*** *with branch sets A (upstairs) and B (downstairs) if the following conditions are satisfied:*
(i) The preimages of open sets of N under f forms a basis for the topology of M
(ii) $f(A) = B$ and $f(M \setminus A) = N \setminus B$
(iii) Every point in $N \setminus B$ has a neighborhood U such that each component of $f^{-1}(U)$ is mapped homeomorphically onto U under f.

Also, the restriction map $f : M \setminus A \to N \setminus B$ is called ***the associated unbranched covering****. The branch point $a \in A$ is said to have* ***branching index*** *k if f is a degree k-map around the point a.*

Remark 2.4.1. *Given a manifold N, the branch set B and a finite branched covering of $N \setminus B$, the total space M can be determined. However, M does not always have to be a manifold. (Fox,1957)*

Example 2.4.1. *Let $\mathbb{C}^2$ be the complex sphere. Then the map $z \mapsto z^k$ on $\mathbb{C}^2$, which maps the complex unit disc to itself k-times, is an example of a branched covering with branched index k at the origin.*

2.4.1 Cyclic Branched Covers of S^3

In this section we will construct cyclic branched covers, branched over a knot.

Definition 2.4.2. *Let K be a knot in S^3 and $n \geq 2$ be an integer. Then the n- fold cyclic cover of S^3 branched along the knot K is called* ***n-fold cyclic branched cover*** *of K.*

Let us first construct the n-fold cyclic cover of $S^3 \setminus K$:

Let T be the unknotted solid torus in S^3 in the Figure 2.11. Then by a homeomorphism $h : S^3 \setminus int(T) \to S^3 \setminus int(T)$ we can give a twist to the interior $int(T)$ of the

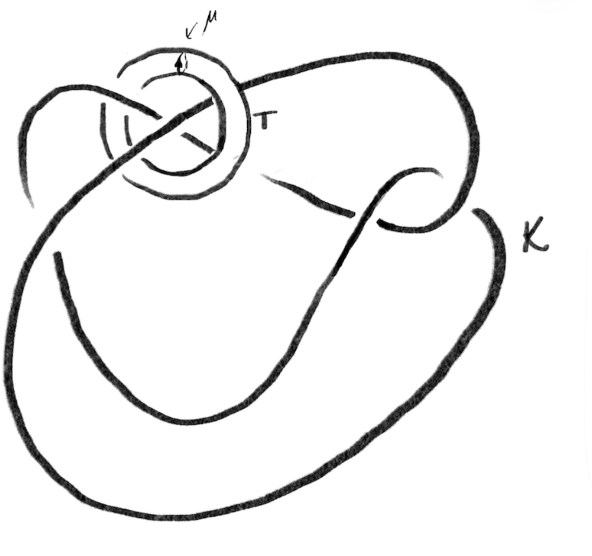


Figure 2.11: construction of a cyclic branched cover over a knot -1- [14]

torus T (Figure 2.12).

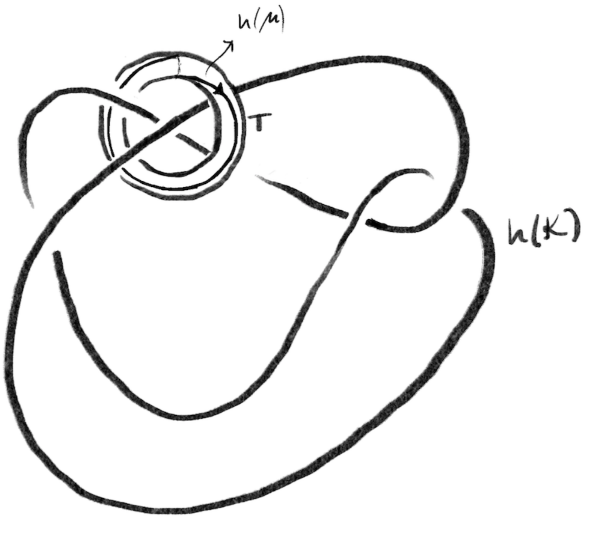


Figure 2.12: construction of a cyclic branched cover over a knot -2- [14]

In order to obtain Figure 2.13, trace the shape of the knot with the twisted torus. With this process, $h(K)$ becomes homeomorphic to S^1 and the torus T has the knot type of K, where the knot $h(\mu)$ lies on T.

The Figure 2.13 lies in an open solid torus. Then the complement of $S^3 \backslash K$ is obtained by attaching an open solid torus to that open solid torus where $h(\mu)$ lies in. The two

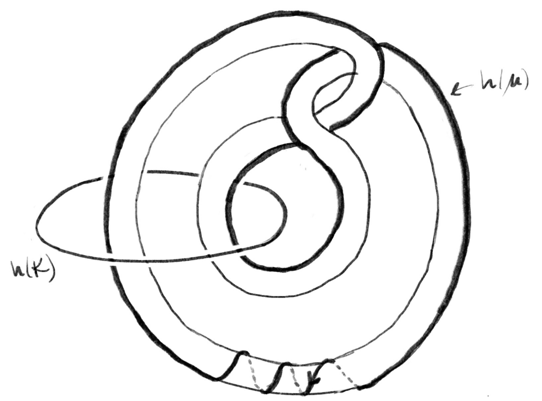


Figure 2.13: construction of a cyclic branched cover over a knot -3- [14]

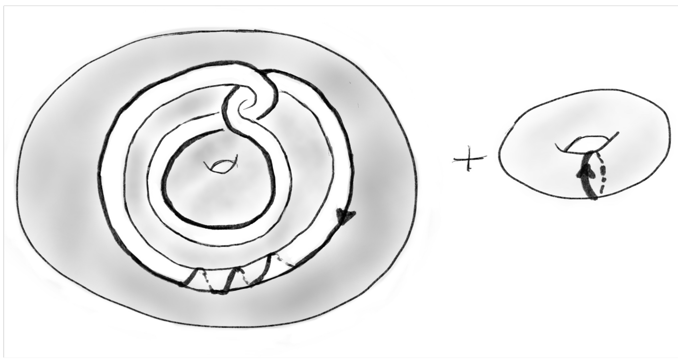


Figure 2.14: construction of a cyclic branched cover over a knot -4- [14]

solid tori is attached by gluing the meridian curve and the curve $h(\mu)$ to each other as shown in Figure 2.14.

The n- fold cyclic cover X_n of $X_n = S^3 - K$ is given by sewing n solid tori to $h(\mu_i)$'s from their meridian curves as in the following Figure 2.15. (for $n = 3$)

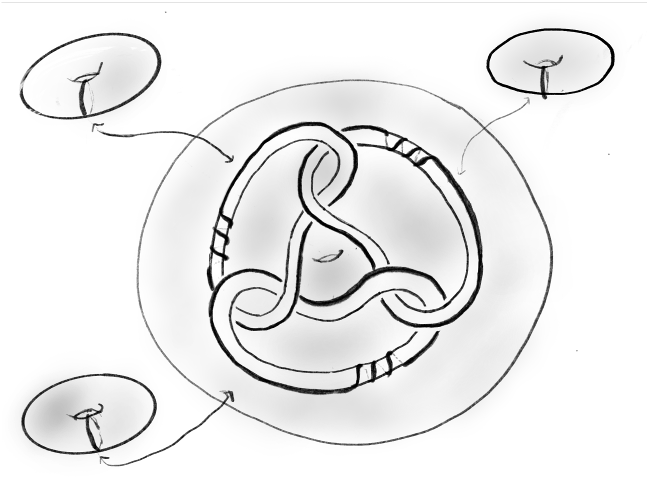


Figure 2.15: n-fold cyclic cover for $n = 3$[14]

Now, we will see how to construct an n-fold cyclic branched cover: Let N be an open tubular neighborhood of the knot K. Then consider the n-fold cyclic unbranched cover of $S^3 - N$. Note that the boundary of N is a torus. Moreover, in the preimage of the covering map, the meridian on ∂N is wrapped n-times to itself, while the longitude has n-distinct loops upstairs. (Figure 2.16)

Along the boundaries, attach a solid torus $S^1 \times D^2$ to this unbranched cover such that a meridian $* \times S^1$ matches by the preimage of a meridian of ∂N. This gives a closed connected manifold M^3. Then the covering map $M^3 \to S^3$ can be expended to a branched covering by mapping $D^2 \times S^1$ to N with the product map $(z \mapsto z^n/|z^{n-1}|) \times id$ since $N \cong D^2 \times S^1$. Then the knot K will be the branch set downstairs.

Example 2.4.2. *The* 3*-fold cyclic branched covering of* S^3 *branched along the figure eight knot is given by Figure 2.17.*

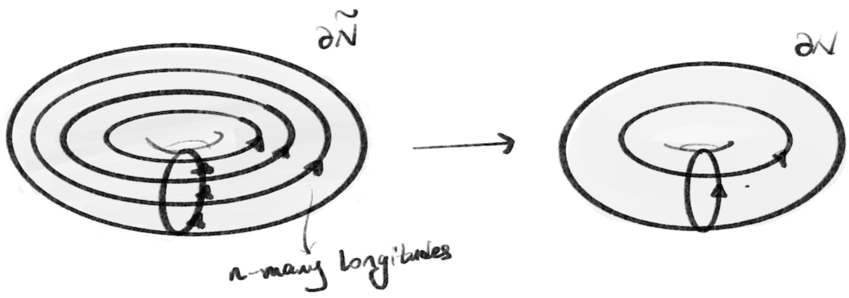


Figure 2.16: construction of an n-fold cyclic branched cover [14]

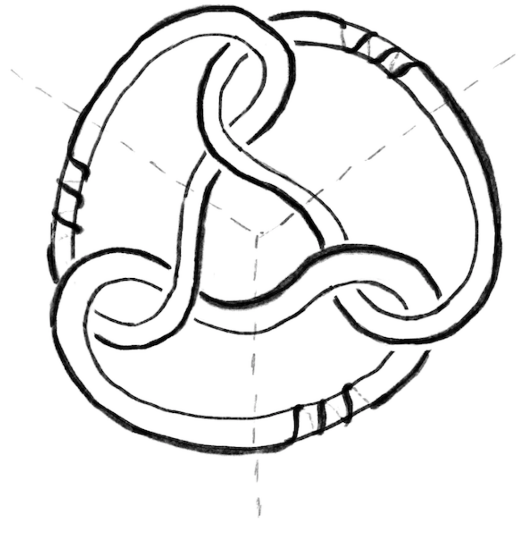


Figure 2.17: 3-fold cyclic branched covering of S^3 along the figure-eight knot [14]

2.5 A Short Review of Surgery Theory

In this section, we will introduce Dehn surgery and how to perform it. The content will be based on Chapter 9 of [14].

Definition 2.5.1. *Let the following information be given:*

(i) a 3-manifold M,

(ii) a link L in the interior of M such that $L = L_1 \cup ... \cup L_n$ where L_i's are simple closed curves,

(iii) closed and disjoint tubular neighborhoods N_i's of L_i's in $int(M)$,

(iv) a simple closed curve J_i on the boundary ∂N of each N.

Then the manifold $\tilde{M}$ can be constructed as

$$\tilde{M} := (M \setminus (int(N_1) \cup ... \cup int(N_n))) \bigcup_h (N_1 \cup ... \cup N_n)$$

*such that h is the union of homeomorphisms $h_i : \partial N_i \to \partial N_i$ where $h_i(\mu_i) = J_i$ for a meridian curve μ_i of N_i. The 3-manifold $\tilde{M}$ is said to be constructed by a **Dehn surgery** on M along the link L with surgery instructions (iii) and (iv).*

2.5.1 Surgery Instructions in S^3

[14] Let L be an oriented link. Give the orientation of L_i to the longitude λ_i of the tubular neighborhood N_i. Also, the linking number of the meridian μ_i with the component L_i is $+1$. Then $\{\mu_i, \lambda_i\}$ gives a basis so that the curve J_i is given by $h_*(\mu_i) = [J_i] = a_i\lambda_i + b_i\mu_i$ where b_i is the linking number $lk(L_i, J_i)$.

Definition 2.5.2. *Let $r_i = b_i/a_i$. Then the ratio r_i is called the surgery coefficient associated to L_i. If $a_i = 0$ then $r_i = \infty$. A surgery with a rational surgery coefficient is called **rational surgery**. In the case of $a_i = \pm 1$, it is called **integral surgery**.*

A rational number associated to the components of the link L in S^3 determines a surgery and hence a closed oriented 3-manifold.

Theorem 2.5.1. ***(Lickorish and Wallace)*** *[15] Let M be a closed orientable 3-manifold. Then M can be obtained by an integral surgery on S^3 along the link $L \subset S^3$.*

Lemma 2.5.1. *Let H and H' be two handlebodies and let $h_1, h_2 : \partial H \to \partial H'$ be two homeomorphisms on the boundary surfaces of the handlebodies. Also let $h_1 = h_2\tau_c$ be a twist along a simple closed curve $c \subset \partial H$. Then by performing integral surgery to the manifold $M = H \cup_{h_1} H'$ along a knot $K \in M'$, which is isotopic to the curve c, one can obtain the manifold $M' = H \cup_{h_2} H'$.*

Proof. [15] Consider the translation of closed curve c on ∂H as it is lying inside the handlebody. This gives a knot $k \subset H$. Let us denote the tubular neighborhood of the knot k by $N(k)$ and the annulus connecting the curve c and $\partial N(k)$ by $A \cong S^1 \times I$, as in Figure 2.18.

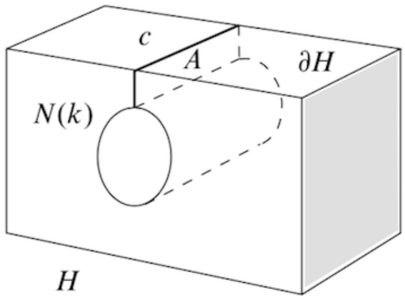


Figure 2.18: on Lemma 2.5.1[15]

Define a homeomorphism $\psi : H \backslash N(k) \to H \backslash N(k)$ which opens the space $H \backslash N(k)$ along A and rotates one of the edges by 360° and attach them together. Then $\psi|_{\partial H} = \tau_c$ and $\psi|_{\partial N(k)}$ gives a twist along $A \cap N(k)$. Define ϕ a homeomorphism from $M_2' = (H \setminus N(k)) \cup_{h_2} H'$ to $M_1' = (H \setminus N(k)) \cup_{h_1} H'$ which is given by

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x \in H \setminus N(k) \\ x & \text{if } x \in H' \end{cases}$$

Note that $\psi|_{\partial H} = \tau_c$ and $h_1 = h_2\tau_c$. Therefore, on the boundary points one has $x \in \partial H \setminus N(k) \cap H'$ $\phi(x) = x$. So, ψ is well defined on the boundary. (Figure 2.19)

Hence, by removing the solid tori $N(k)$ from the manifolds M_1 and M_2, the remaining

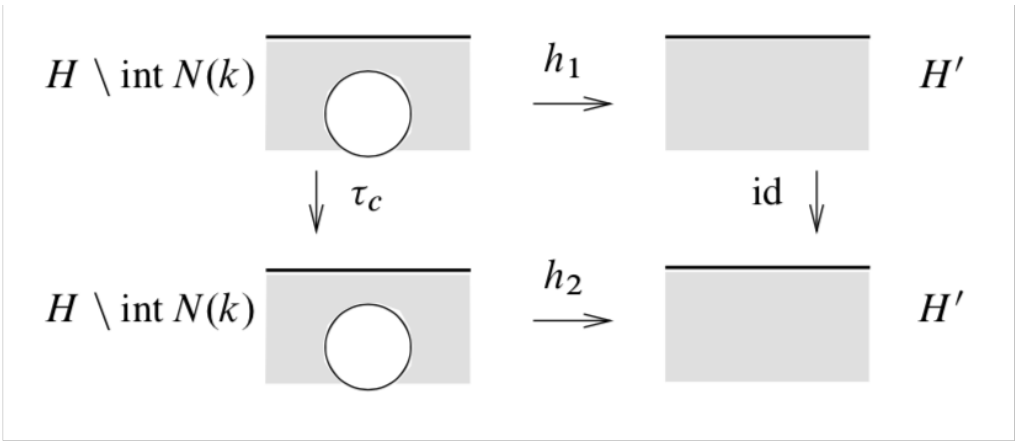


Figure 2.19: on Lemma 2.5.1[15]

manifolds become homeomorphic. So, M_2 can be obtained from M_1 by performing a surgery along the knot k. If m is the meridian of the boundary torus $\partial N(k)$, then $\phi(m) = m \pm l$. Hence, the surgery is an integral surgery.

□

Proof. (of Theorem 2.5.1) [15] Every manifold M has a Heegaard splitting $M = H \cup_{h_2} H^{'}$ where h_2 is an orientation-reversing homeomorphism between the bounderies of the handlebodies H and $H^{'}$ of genus g. In particular let $S^3 = H \cup_{h_1} H^{'}$ be the Heegaard splitting of S^3. Then $h_2^{-1} h_1 : \partial H \to \partial H$ is an orientation preserving homeomorphism. Therefore, $h_2^{-1} h_1$ can be considered as the composition of twists along the curves C_i, i.e. $h_2^{-1} h_1 = \tau_{c_1} \tau_{c_2} ... \tau_{c_n}$. The Lemma 2.5.1 implies that performing integral surgery along a knot is the same as applying a Dehn twist with gluing homeomorphism. Therefore, applying a sequence of twists along knots gives a sequence of surgeries along knots, and hence a surgery on a link.

□

Definition 2.5.3. ***(Lens spaces)*** *Let $p \in \mathbb{Z}^+, q \in \mathbb{Z} \setminus 0$ be relatively prime. Let a $\mathbb{Z}/p$ action on $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ is given by $\phi_{(p,q)} : S^3 \to S^3$ such that $\phi_{(p,q)}(z, w) = (e^{2\pi i/p} z, e^{-2\pi q i/p} w)$. The oriented 3-manifold given by the quotient space $L(p, q) := S^3/(z, w) \sim \phi_{(p,q)}(z, w)$ is called a* ***lens space****.*

Proposition 2.5.1. *The fundamental group of a lens space $L(p, q)$ is $\pi_1(L(p, q)) =$*

$\mathbb{Z}_p$.

Proof. The projection map $p : S^3 \to S^3/\mathbb{Z}_p \cong L(p,q)$ is a covering map. Note that S^3 is path connected. Therefore by Proposition 1.40 in [6], the group of deck transformations of this covering map is $\pi_1(S^3/\mathbb{Z}_p) = \pi_1(L(p,q)) = \mathbb{Z}_p$.

$\square$

Example 2.5.1. *The lens space $L(p,q)$ can be obtained by p/q-surgery over the trivial knot in S^3:*

First remove the tubular neighborhood T of the trivial knot from S^3. Then since T is unknotted, $\overline{S^3 \setminus T}$ is also an unknotted solid torus. Next, sew T back to $\overline{S^3 \setminus T}$ via the homeomorphism $h : \partial T \to \partial T$, $h(m) = ql + pm$ where m, l is meridian and longitude of the torus respectively. Then the meridian curve $(0,1)$ will be attached to the torus knot (p,q). Then the manifold obtained from this surgery is the lens space $L(p,q)$.

2.6 Seifert Manifolds

This section is based on Chapter 1.6 of [15].

Remove the interiors of n disjoint discs from a 2-sphere, i.e. let $F := S^2 \setminus (int(D_1^2) \cup ... \cup int(D_n^2))$. Then the manifold $F \times S^1$ is a compact orientable 3-manifold with boundary and $\partial(F \times S^1) = (\partial D_i^2) \times S^1$, $i = 1, ..., n$.

Note that the fundamental group of $F \times S^1$ is $\pi_1(F \times S^1) =< x_1, ..., x_n, h \mid hx_i = x_i h \ , \ x_1...x_n = 1 >$ where the generators x_i represent the boundary curves of n-discs.

Let n pairs of integers $(a_1, b_1), ..., (a_n, b_n)$ be given such that a_i and b_i are relatively prime and $a_i \geq 2$ for all n. Then glue a solid torus to $\partial(D_i) \times S^1$ for all $1 \leq i \leq n$ so that its meridian is glued to a curve isotopic to $a_i.x_i + b_i.h$.

For all i, the image curve of $\{0\} \times S^1$ after the gluing process is called the i-th **singular fiber**.

Definition 2.6.1. *The resulting closed manifold with this construction is called the* ***Seifert manifold*** $M((a_1, b_1), ..., (a_n, b_n))$ *of genus* 0 *with* n *singular fibers [15].*

Example 2.6.1.

- *The lens space* $L(b, a)$ *is a Seifert manifold* $M(a, b)$ *with one singular fiber.*

- *The lens space* $L(a_1b_2 + a_2b_1, a_1a_2)$ *is a Seifert manifold* $M((a_1, b_1), (a_2, b_2))$ *with two singular fibers [15].*

Remark 2.6.1. *The Figure 2.20 gives a rational surgery description for a Seifert manifold* $M((a_1, b_1), ..., (a_n, b_n))$.

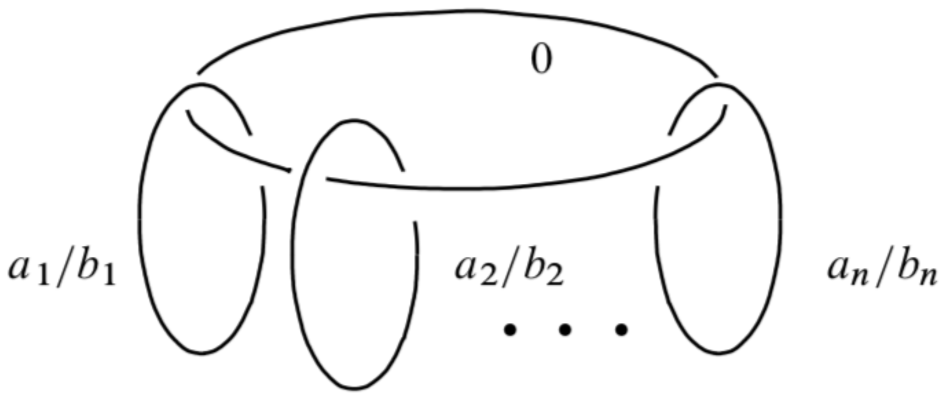


Figure 2.20: Surgery description of a Seifert manifold [15]

2.7 Orbifolds

We will give a formal definition of an orbifold in terms of its local structure. Intuitively, an orbifold can be seen locally as a manifold which has some locally non-manifold points.

Definition 2.7.1. *An* ***orbifold chart*** *on an* n*-dimensional topological space* X *is the triple* $(\tilde{U}, G, \pi)$ *satisfying the followings:*

- $\tilde{U}$ *is an open subset of* $\mathbb{R}^n$

- G *is finite diffeomorphism group of* $\tilde{U}$

- *$\pi : \tilde{U} \to X$ is a smooth map which makes the below diagram commute: (Here, p is the orbit map and $\bar{\pi} : \tilde{U} \to X$ is a homeomorphism onto an open subset $U \subset X$.)*

$$\begin{tikzcd} \tilde{U} \arrow[r, "p"] \arrow[dr, "\pi"'] & \tilde{U}/G \arrow[d, "\bar{\pi}"] \\ & X \end{tikzcd}$$

*Moreover, suppose that two orbifold charts $(\tilde{U}_1, G_1, \pi_1)$ and $(\tilde{U}_2, G_2, \pi_2)$ on X and a point $x \in U_1 \cap U_2 = \pi_1(\tilde{U}_1) \cap \pi_2(\tilde{U}_2)$ are given. If there is an open neighborhood U of x with $U \subset U_1 \cap U_2$ and a chart $(\tilde{U}, G, \pi)$ with $U = \pi(\tilde{U})$ which induces an embedding on each chart $(\tilde{U}, G, \pi) \hookrightarrow (\tilde{U}_i, G_i, \pi_i)$, then these orbifold charts are said to be **compatible**.*

Definition 2.7.2. *The set of compatible charts $\mathcal{U} = \{(\tilde{U}_\alpha, G_\alpha, \pi_\alpha)\}_{\alpha \in I}$ on X is called an **orbifold atlas** on X.*

Definition 2.7.3. *A paracompact, Hausdorff topological space $X_{\mathcal{O}}$ with a compatible orbifold atlas $\mathcal{U}$ on it is called an **orbifold**, denoted by $\mathcal{O}$. The topological space $X_{\mathcal{O}}$ is called the underlying topological space of the orbifold $\mathcal{O}$.*

The following proposition provides a way to obtain an orbifold from a manifold.

Proposition 2.7.1 ([19])**.** *If the action of a group G on a manifold M is properly discontinuous, then M/G is an orbifold.*

Proof. We will find a cover for M/G by using the manifold structure of M. For each point $x \in M$, there is a corresponding point $\tilde{x} \in M/G$ such that $\pi(x) = \tilde{x}$, coming from the projection map $\pi : M \to M/G$. Let G_x be the isotropy subgroup of x. Now take a neighborhood of x, say U_x, such that the action of G_x keeps U_x invariant and $U_x \cap h.U_x = \emptyset$ when $h \in G \setminus G_x$. This is possible since the action is properly discontinuous. Then under the map $\pi : M \to M/G$, we have $\pi(U_x) = U_x/G_x$ and this U_x/G_x forms a neighborhood for $\tilde{x}$ in M/G. So, $\{U_x/G_x\}$ is a cover for M/G, which is homeomorphic to a quotient of some Euclidean space by the group action. Now, we need to say that nonempty finite intersections belong to that cover also. Consider some finite intersection $U_{x_1}/G_{x_1} \cap U_{x_2}/G_{x_2} \cap ... \cap U_{x_k}/G_{x_k} \neq \emptyset$. This means that there are some elements $g_{x_1}, g_{x_2}, ..., g_{x_k} \in G$ such that $g_{x_1}.U_{x_1} \cap g_{x_2}.U_{x_2} \cap ... \cap g_{x_k}.U_{x_k} \neq \emptyset$. Then since the action is properly continuous, the isotropy group can be taken as $g_{x_1}G_{x_1}g_{x_1}^{-1} \cap g_{x_2}G_{x_2}g_{x_2}^{-1} \cap ... \cap g_{x_k}G_{x_k}g_{x_k}^{-1}$ so that the quotient of the finite intersection by this group belongs to the cover $\{U_x/G_x\}$. Hence, M/G is an orbifold.

□

Not all but many orbifolds can be obtained by the action of some finite group on a manifold. In this section, we will consider only the ones obtained by the quotient of a manifold by a finite group action.

Definition 2.7.4. *The set of points on an underlying space of an orbifold which have nontrivial isotropy subgroup is called the* ***singular locus*** *or the* ***branch set****.*
It is denoted by $\Sigma(\mathcal{O}) = \{x \in X_{\mathcal{O}} \mid G_x \neq e\}$

Example 2.7.1. *An orientation-preserving $\mathbb{Z}_2$ action on a torus is a 2-orbifold, called a pillowcase.*
This action corresponds to π-rotation of the torus through the horizontal axis. Under this action, each point on torus is mapped to some other point, except the four points which lie on the axis of rotation. Those fixed points form the singular locus of the orbifold. Also, one can see that S^2 is the underlying topological space of this orbifold. [3]

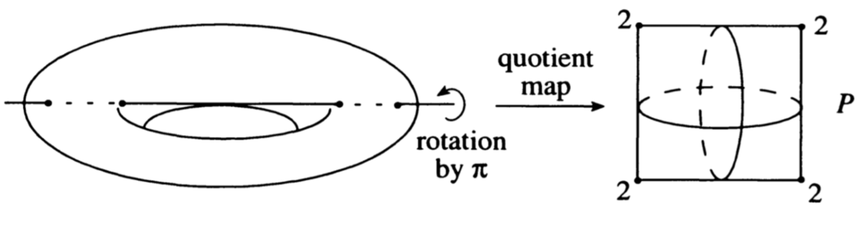


Figure 2.21: pillowcase [3]

Definition 2.7.5. *The* ***Euler characteristic*** $\chi(\mathcal{O})$ ***of an orbifold*** $\mathcal{O}$ *is given by*

$$\chi(\mathcal{O}) = \sum_{\sigma \in \mathcal{O}} (-1)^{dim(\sigma)} \frac{1}{|\Gamma(\sigma)|}$$

where $\Gamma(\sigma)$ is the group assigned to each cell σ of $\mathcal{O}$ [3].

Fact 1. *Let G be a finite subgroup of $SO(3)$. Then G can be the cyclic group $\mathbb{Z}_n$, the dihedral group D_{2n}, the symmetric group S_4 or the alternating groups A_4 and A_5.*

One can see a detailed proof of this fact in Chapter 19 of [2].

2.7.1 Orbifold coverings

Definition 2.7.6. *A map $f : \tilde{\mathcal{O}} \to \mathcal{O}$ is called an* ***orbifold covering*** *if the projection map $p : X_{\tilde{\mathcal{O}}} \to X_{\mathcal{O}}$ is continuous and if the following conditions are satisfied:*

- *For every point $x \in X_{\mathcal{O}}$, there is a neighborhood $U = \tilde{U}/G$ in $X_{\tilde{\mathcal{O}}}$.*
- *For every component V_i of $f^{-1}(U)$, there is a subgroup G_i of G such that $V_i = \tilde{U}/G_i$.*

Example 2.7.2. *Let M be a manifold and G be a group acting properly discontinuously on M. Then for a subgroup G_i of G, the map $M/G_i \to M/G$ gives an orbifold covering.*

Example 2.7.3. *The Example 2.7.1 gives an orbifold covering between the torus T^2 and the resulting orbifold under $\mathbb{Z}_2$ action, say $S^2(2,2)$. Moreover, this orbifold covering $T^2 \to S^2(2,2)$ is actually a 2-fold covering.*

The following proposition is given as an exercise in [3]. Here, we provide a solution to that exercise and represent it as a proof of the proposition.

Proposition 2.7.2. *(i) If the orbifold $\mathcal{O}$ has a k-fold orbifold covering $\tilde{\mathcal{O}} \to \mathcal{O}$, then $\chi(\tilde{\mathcal{O}}) = k\chi(\mathcal{O})$.*

(ii) If $\mathcal{O}$ is a 2-orbifold which has branch points of order m_i, then the Euler characteristic of $\mathcal{O}$ can be computed by the following equality:

$$\chi(\mathcal{O}) = \chi(X_{\mathcal{O}}) - \sum_i (1 - 1/m_i)$$

Proof. *Part (i): Let $f : \tilde{\mathcal{O}} \to \mathcal{O}$ be a k-fold orbifold covering. We know that the Euler characteristic of $\mathcal{O}$ is $\chi(\mathcal{O}) = \sum(-1)^{dim\sigma}\frac{1}{|\Gamma_\sigma|}$. Consider the restriction of the covering to each cell σ. As f is an orbifold covering, for each $\sigma \in \tilde{U}/\Gamma_\sigma$ there is some $\tilde{U}/\Gamma_{\sigma_i}$ where Γ_{σ_i} is a subgroup of Γ_σ. Now, the restriction map $f|_\sigma : \tilde{U}/\Gamma_{\sigma_i} \to \tilde{U}/\Gamma_\sigma$ is also a k-fold cover. This means that $\frac{|\Gamma_\sigma|}{|\Gamma_{\sigma_i}|} = k$. Hence, $\chi(\tilde{\mathcal{O}}) = \sum(-1)^{dim\sigma}\frac{1}{|\Gamma_{\sigma_i}|} = \sum(-1)^{dim\sigma_i}\frac{1}{|\Gamma_\sigma|/k} = \sum(-1)^{dim\sigma}\frac{k}{|\Gamma_\sigma|} = k.\sum(-1)^{dim\sigma}\frac{1}{|\Gamma_\sigma|} = k\chi(\mathcal{O})$.*

Part (ii): Let σ_j and γ_k represent a 1-cell and 2-cell of the underlying topological space of the orbifold. If we see the branch points as 0-cells, then the Euler characteristic of the underlying space becomes $\chi(X_\mathcal{O}) = \sum_i 1 - \sum_j \sigma_j + \sum_k \gamma_k$. On the other hand, the groups associated to 1-cells and 2 cells are trivial as they are not in the singular locus of $\mathcal{O}$. So by the Definition 2.7.5, the Euler characteristic of $\mathcal{O}$ is $\chi(X_\mathcal{O}) = \Sigma_i 1/m_i - \Sigma_j\sigma_j + \Sigma_k\gamma_k$. Now, by subtracting the second equality from the first one, we obtain that $\chi(\mathcal{O}) = \chi(X_\mathcal{O}) - \sum_i(1 - 1/m_i)$.

Proposition 2.7.3. *For every orbifold $\mathcal{O}$ there is a universal cover $\pi : \tilde{\mathcal{O}} \to \mathcal{O}$. This means that there are base points $x \in X_\mathcal{O} \setminus \Sigma(\mathcal{O})$ and $\tilde{x} \in X_{\tilde{\mathcal{O}}} \setminus \Sigma(\mathcal{O})$ with $\pi : (\tilde{\mathcal{O}}, \tilde{x}) \to (\mathcal{O}, x)$ so that $\pi(\tilde{x}) = x$. Moreover, for some other orbifold covering $\pi^{'} : (\tilde{\mathcal{O}}^{'}, \tilde{x}^{'}) \to (\mathcal{O}, x)$ with $\pi^{'}(\tilde{x}^{'}) = x$, the map $p : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}^{'}$ is also an orbifold covering with $p(\tilde{x}) = \tilde{x}^{'}$.*

$$\begin{tikzcd} & \tilde{\mathcal{O}} \arrow[dl, "p"'] \arrow[dd, "\pi"] \\ \tilde{\mathcal{O}}' \arrow[dr, "\pi'"'] & \\ & \mathcal{O} \end{tikzcd}$$

One can see a proof of this in Chapter 13 of [19].

2.7.2 Fundamental group of an orbifold

Definition 2.7.7. *The **fundamental group** $\pi_1^{orb}(\mathcal{O})$ of an orbifold $\mathcal{O}$ is defined to be the deck transformation group of the universal cover $\tilde{\mathcal{O}}$ of the orbifold $\mathcal{O}$.*

Remark 1. *For a two dimensional orbifold $\mathcal{O}$, the fundamental group of $\mathcal{O}$ can be found by $\pi_1^{orb}(\mathcal{O}) = \pi_1(\mathcal{O} \setminus \Sigma(\mathcal{O}))/G$ if G is acting properly discontinuously on $\mathcal{O}$.*

Fact 2. *The free product with amalgamation of two groups A and B is the quotient group $(A * B)/N$ where N is the smallest nontrivial normal subgroup of the free product $A * B$.*

Theorem 2.7.1. ***(Van Kampen Theorem for orbifolds)*** *Let the orbifold $\mathcal{O}$ be the union of two orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$, i.e. $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, such that the intersection $\mathcal{O}_1 \cap \mathcal{O}_2$ is path connected. Then the fundamental group of $\mathcal{O}$ is given by the amalgamated free product of $\mathcal{O}_1$ and $\mathcal{O}_2$.*

$$\pi_1^{orb}(\mathcal{O}) = \pi_1^{orb}(\mathcal{O}_1) *_{\pi_1^{orb}(\mathcal{O}_1 \cap \mathcal{O}_2)} \pi_1^{orb}(\mathcal{O}_2)$$

Example 2.7.4. *Let $\mathcal{O}$ be the n-teardrop orbifold over $S^2 = D_1 \cup D_2$ and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the corresponding orbifolds over disks D_1 and D_2. $\mathcal{O}_1$ is a cone, and $\pi_1^{orb}(\mathcal{O}_1) \cong \mathbb{Z}_n$ and $\mathcal{O}_2$ is just a disk, hence simply-connected. Moreover, $\mathcal{O}_1 \cap \mathcal{O}_2$ is an annulus, thus the $\pi_1^{orb}(\mathcal{O}_1 \cap \mathcal{O}_2) \cong \mathbb{Z}$. The map induced by inclusion of $\mathcal{O}_1 \cap \mathcal{O}_2$ in $\mathcal{O}_1$ is surjective. So by the Seifert–Van Kampen theorem we have $\pi_1^{orb}(\mathcal{O})$ is trivial.*

2.7.3 Classification of some two and three dimensional orbifolds

Definition 2.7.8. *Let G be a finite group acting on an n- disc B^n by orientation-preserving homeomorphisms. Then the resulting orbifold B^n/G is called a **discal orbifold**. Similarly, if G acts on an n-sphere S^n then S^n/G is called a **spherical orbifold**.*

The following two theorems are about the classification of spherical 2-orbifolds and discal 3-orbifolds when the action is orientation-preserving. It is a well-known fact that a finite group of orientation-preserving homeomorphisms is isomorphic to finite subgroups of $SO(3)$ [24]. Therefore, the group G in those theorems are immediately considered to be finite subgroups of $SO(3)$.

Theorem 2.7.2. *Let G be a finite subgroup of $SO(3)$. Assume that G acts on the sphere S^2. Also let $F(n_1, n_2, ..., n_k)$ denote the 2-orbifold whose underlying topological space is the surface F and n_i's are the order of groups each point in singular locus. Then for each group G, the 2-orbifold S^2/G is one of the following:*

- *If $G = \mathbb{Z}_n$ then $S^2/G = S^2(n, n)$.*

- *If* $G = D_{2n}$ *then* $S^2/G = S^2(2, 2, n)$.
- *If* $G = A_4$ *then* $S^2/G = S^2(2, 3, 3)$.
- *If* $G = S_4$ *then* $S^2/G = S^2(2, 3, 4)$.
- *If* $G = A_5$ *then* $S^2/G = S^2(2, 3, 5)$.

Proof. We will show that the stated orbifolds satisfy the given group actions by using their orbifold covering and Euler characteristic. We will show only two cases. The rest can be shown similarly.

- For the first case, the map $\pi : S^2 \to S^2(n, n)$ is an n-fold orbifold covering. By using part (i) of the Proposition 2.7.2, one has $\chi(S^2) = n\chi(S^2(n, n))$. Since $\chi(S^2) = 2$, this implies that $\chi(S^2(n, n)) = \frac{2}{n}$. On the other hand, by the Part (ii) of Proposition 2.7.2 the Euler characteristic of $S^2(n, n)$ can be computed as $\chi(S^2(n, n)) = \chi(S^2) - ((1 - 1/n) + (1 - 1/n)) = 2 - \frac{2n-2}{n} = \frac{2}{n}$ as desired.
- For the last case, the 2-sphere S^2 is a 60-fold cover of the orbifold $S^2(2, 3, 5)$ since $|A_5| = 60$. Again since the underlying space of the orbifold is S^2, Proposition 2.7.2 implies that $\chi(S^2(2, 3, 5)) = \chi(S^2) - ((1 - 1/2) + (1 - 1/3) + (1 - 1/5)) = 1/30$. Observe that part (i) of the same proposition gives the same result: $\chi(S^2(2, 3, 5)) = \chi(S^2)/60 = 1/30$.

□

Theorem 2.7.3. *Consider a discal orbifold* B^3/G *where* G *is a subgroup of* $SO(3)$. *Then the branch set of the orbifold is either a graph with one edge with branch order* n *or a trivalent graph where the possible branch orders of the edges are* $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ *and* $(2, 3, 5)$ *where* $n \geq 2$.

2.7.4 Handlebody Orbifolds

Before introducing handlebody orbifolds, for further use, let us look at a result given in [11].

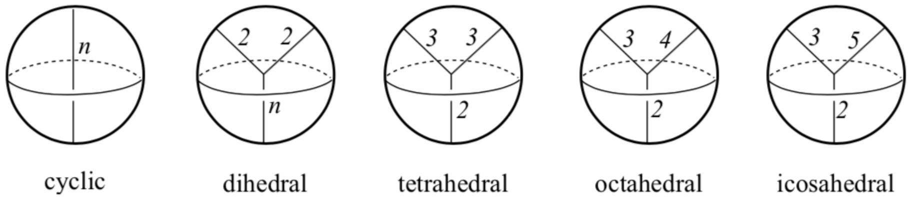


Figure 2.22: 3-discal orbifolds [21]

Theorem 2.7.4. ***(Equivariant Dehn's Lemma for discs)*** *Let a collection of disjoint Jordan curves $\{\gamma_1, ..., \gamma_n\}$ which are on the boundary of a 3-manifold M. Assume that γ_i is homotopically trivial in M for $i = 1, ..., n$. If G is a compact group with a free, orientation-preserving action on $\cup_{i=1}^{n} \gamma_i$, then there are embedded invariant discs $\{D_1, ..., D_n\}$ which are pairwise disjoint such that $\partial D_i = \gamma_i$ and their union is invariant under the action of G [11].*

Let G be a finite group with an orientation-preserving action on a handlebody V_g. Consider a properly embedded 2-disc in V_g such that $\partial D = D \cap \partial V_g$ is a nontrivial closed curve on the boundary surface of V_g. The Theorem 2.7.4 implies that for every $x \in G$ we have either $x(D) = D$ or $x(D) \cap D = \emptyset$. Then cut V_g by a set of disjoint discs $G(D)$, which means removing the interior of a G-invariant regular neighborhood of $G(D)$. Note that these are indeed a collection of 1-handles, $D^2 \times [0, 1]$. Hence we get a collection of handlebodies with a G-action on them.

This procedure of cutting along discs gives a collection of disjoint 3-balls with an action of G. Recall that the finite orientation-preserving group actions on 3-ball are finite subgroups of the orthogonal group $SO(3)$, see Theorem 2.7.3. The possible quotient orbifolds are listed in the Figure 2.22. Hence we have the following definition [23]:

Definition 2.7.9. *Let a finite group G act orientation-preservingly on the handlebody V_g. Then the quotient orbifold $\mathcal{H} := V_g/G$, which are quotients of 3-balls by finite groups of homeomorphisms and connected by cyclic quotients of 1-handles respecting the orders of their singular axes is called a* ***handlebody orbifold****. [23]*

Also, the Figure 3.1 gives an example of how a handlebody orbifold is obtained by respecting the orders of the singular axes.

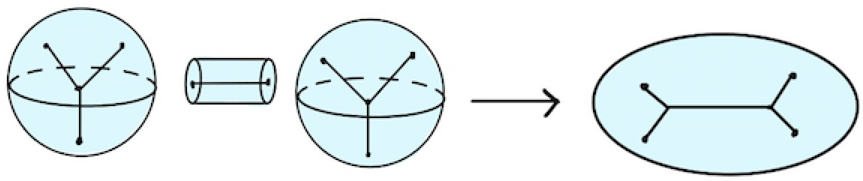


Figure 2.23: obtaining a handlebody orbifold from quotients of 3-balls and a 1-handle

Proposition 2.7.4. *The quotients of handlebodies by finite group actions are exactly the handlebody orbifolds [23].*

2.8 Orbifolds in Dunbar's List

In his paper [4], William Dunbar classifies the geometric 3-orbifolds of the form M/Γ where M is a simply connected 3-space in Thurston's eight geometries and where $\Gamma < Isom(M)$ acts properly discontinuously.

In his list, the orbifolds with indicated underlying space are represented by their singular sets. The singular sets are graphs with valency number at each vertex is at most 3. The edges are labeled by their branching order and the edges with no label are assumed to have branching order 2. The boxes labeled with an integer k represents a k-half twists of two arcs in it. Also, the boxes labeled by two integers m and n represents a rational tangle (m, n) of two arcs. Moreover, that tangle has a strut linking those two arcs to each other and it is labeled by the greatest common divisor of the integers m and n.

Throughout this section, unless otherwise stated, all 3- orbifolds in Dunbar's list are compact, connected orientable orbifolds without boundary.

2.8.1 Seifert Fibered Spherical Orbifolds in Dunbar's List

Definition 2.8.1. • *An orbifold $\mathcal{O}^{n+m}$ is said to be* ***fibered*** *over the base orbifold $\mathcal{B}^n$ with a fiber orbifold $\mathcal{F}^m$ if there is a map $\pi : \mathcal{O}^{n+m} \to \mathcal{B}^n$ such that the inverse image $\pi^{-1}(U)$ of a neighborhood U in $\mathcal{B}^n$ is $\mathbb{R}^n \times \mathcal{F}^m/G$ where G is* *a finite group acting on both $\mathbb{R}^n$ and $\mathcal{F}^m$ such that π is a projection map on $\mathbb{R}^n/G$.*

- *If $\mathcal{F}^m = S^1$ then $\mathcal{O}^{n+1}$ is said to be a **Seifert fibered orbifold** [4].*

Note that a Seifert fibered orbifold $\mathcal{O}^3$ with an empty singular set is a Seifert fibered manifold.

The orbifolds in Table 1 of Figure 2.24 are fibered over 2-sphere. More specifically, they are fibered over 2-orbifolds with underlying space S^2. Next, we will provide those fibering 2-orbifolds for the 3-orbifolds in Table 1 of Dunbar's List.

- Orbifold #01 in Table 1 is fibered over $S^2(n,n)$ where $f, g \geq 1$ are divisors of $n > 1$.
- Orbifold #02 in Table 1 is fibered over $S^2(2,2,n)$ where $f \geq 1$ is a divisor of $n > 1$ and $k \neq 0$
- Orbifolds #03, #04, #05, #06 in Table 1 are fibered over $S^2(2,3,a)$ where $a = 3, 4, 5$.
- Orbifolds #07, #08, #09, #10 in Table 1 are fibered over $S^2(2,3,4)$.
- Orbifolds #11, #12, #13 in Table 1 are fibered over $S^2(2,3,5)$.

Similarly, the orbifolds in Table 2 of Figure 2.25 are fibered over 2-disc. We will provide their fibering orbifolds for the 3-orbifolds in Table 2 of Dunbar's List.

- Orbifold #14 is fibered over $D^2(;)$, $k \neq 0$
- Orbifold #15 is fibered over $D^2(;n,n)$ where $k + m_1/n + m_2/n \neq 0$, $k \neq 0$
- Orbifold #16 is fibered over $D^2(n;)$, $k \neq 0$
- Orbifold #17 is fibered over $D^2(2;n)$ where $k + m/n \neq 0$, $k \neq 0$
- Orbifold #18 is fibered over $D^2(3;2)$ where $k + m/2 \neq 0$, $k \neq 0$
- Orbifold #19 is fibered over $D^2(;2,2,n)$ where $k + m_1/2 + m_2/2 + m_3/n \neq 0$, $k \neq 0$

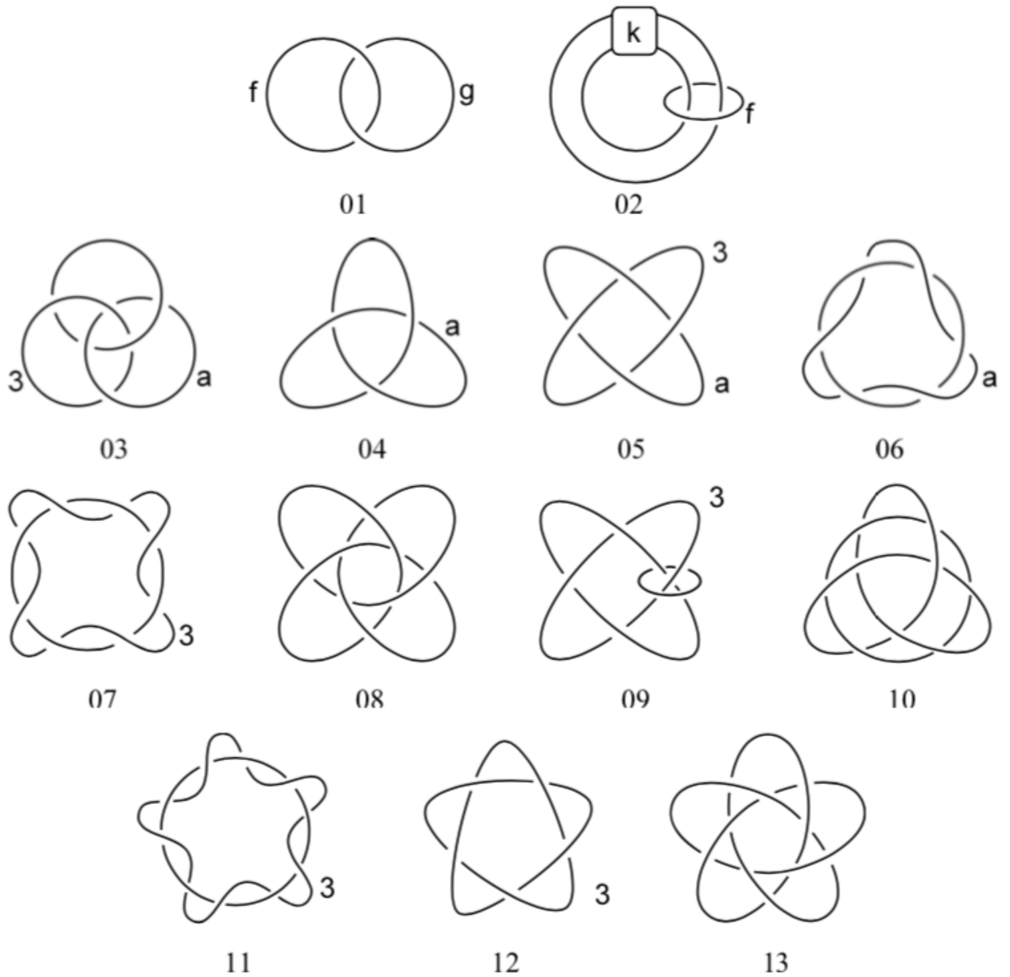


Figure 2.24: **Table 1:** fibered spherical orbifolds fibered over 2-sphere [4],[21]

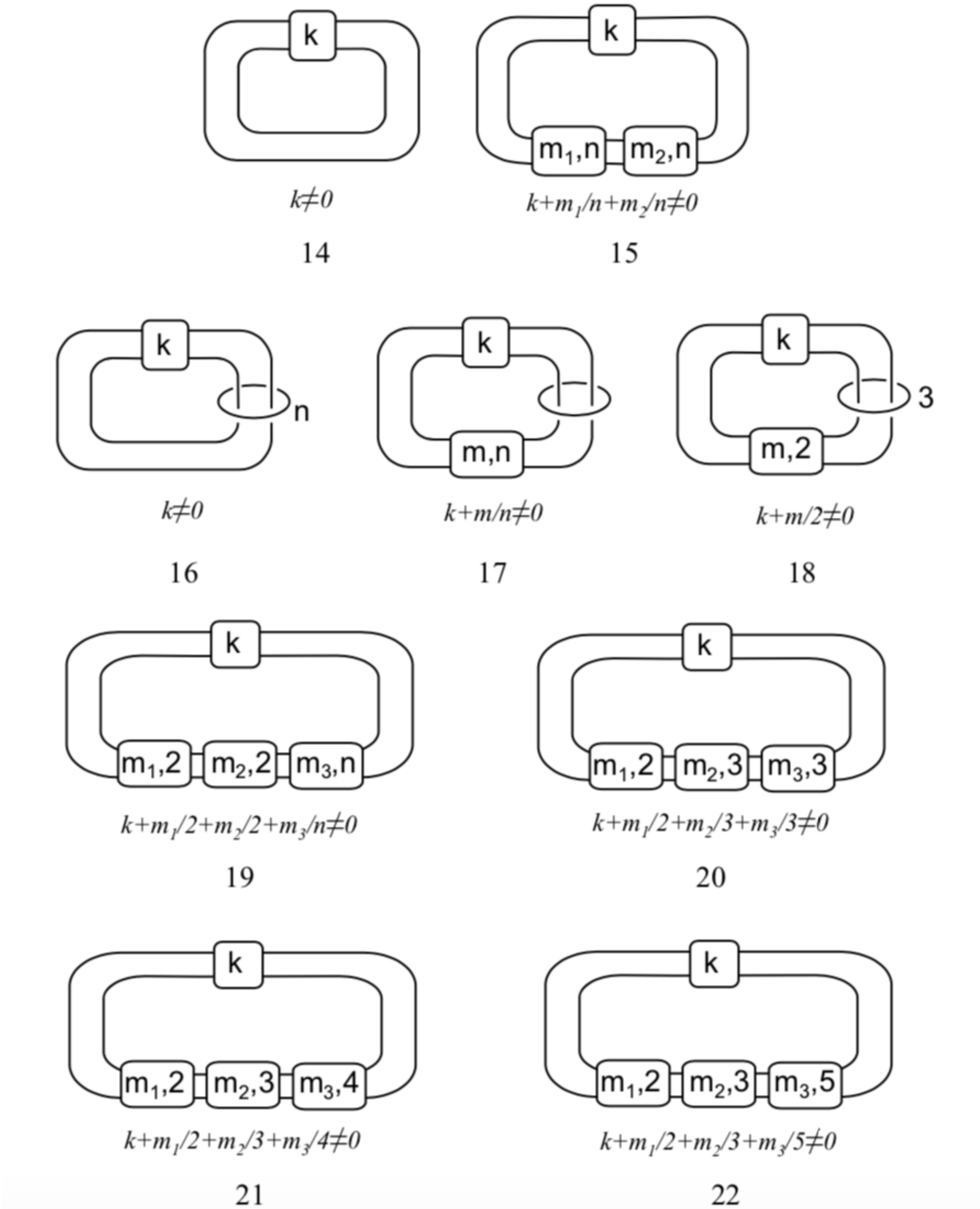


Figure 2.25: **Table 2:** fibered spherical orbifolds fibered over 2-disc [4],[21]

- Orbifold #20 is fibered over $D^2(;2,3,3)$ where $k+m_1/2+m_2/3+m_3/3 \neq 0$, $k \neq 0$
- Orbifold #21 is fibered over $D^2(;2,3,4)$ where $k+m_1/2+m_2/3+m_3/4 \neq 0$, $k \neq 0$
- Orbifold #22 is fibered over $D^2(;2,3,5)$ where $k+m_1/2+m_2/3+m_3/5 \neq 0$, $k \neq 0$

2.8.2 Non-Fibered Spherical Orbifolds in Dunbar's List

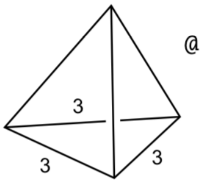
In Table 3 of Figure 2.26 Dunbar classifies the 3-orbifolds with underlying topological space S^3 which are non-fibered.

He uses an algebraic classification theorem of the finite subgroups of $SO(4)$ to determine the groups corresponding to a nonfibered spherical orbifold. Then he folds up the fundamental domains for these group actions on S^3 to obtain their corresponding orbifolds [4].

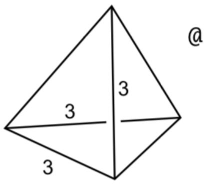
In [4], the fundamental groups of these nonfibered orbifolds are given. In the next chapter we will need the orders of the fundamental groups of these orbifolds. For the sake of completeness, let us explain how these orders are obtained with an example. Let us choose the orbifold #12 given in Figure 2.26l of **Table 3**.

From Figure 2.27, the Wirtinger presentation of this orbifold gives its fundamental group as $\pi_1(\mathcal{O}) =< x, y, z \mid x^2 = y^3 = z^2 = (zy)^2 = (yxz)^2 = (yxzx)^2 = 1 >$.

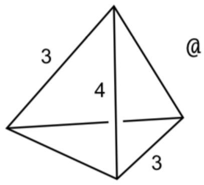
First way of finding the group order is using a computer software. When these generators and their relations are inserted into [GAP], the order of this group is computed as 120. Alternatively, Dunbar finds the fundamental group of this orbifold in [5] as $\pi_1(\mathcal{O}) \cong \mathbf{J} \times_{\mathbf{J}^*} \mathbf{J}$. Since there is a $2:1$ surjection $SO(4) \to SO(3) \times SO(3)$, and since $\pi_1(\mathcal{O})$ is mapped to $J \times_{J^*} J$, we conclude that $|\pi_1(\mathcal{O})| = 2 \times 60 \times 60 \div 60 = 120$.



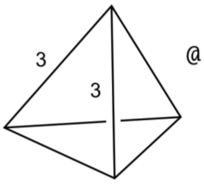
(a) 1



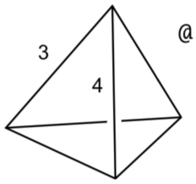
(b) 2



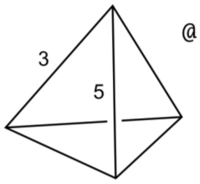
(c) 3



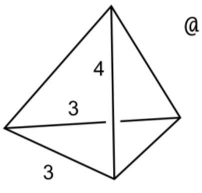
(d) 4



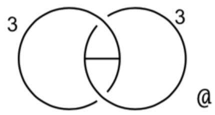
(e) 5



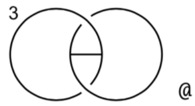
(f) 6



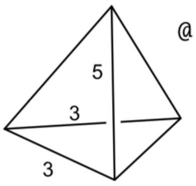
(g) 7



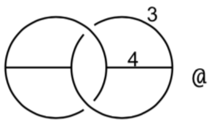
(h) 8



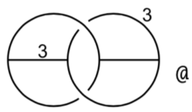
(i) 9



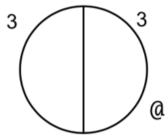
(j) 10



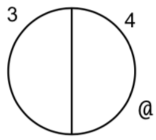
(k) 11



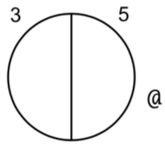
(l) 12



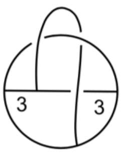
(m) 13



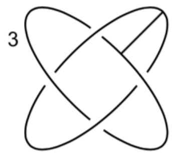
(n) 14



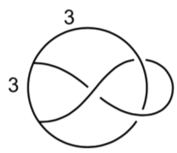
(o) 15



(p) 16



(q) 17



(r) 18

Figure 2.26: **Table 3:** nonfibered spherical orbifolds [5],[21]

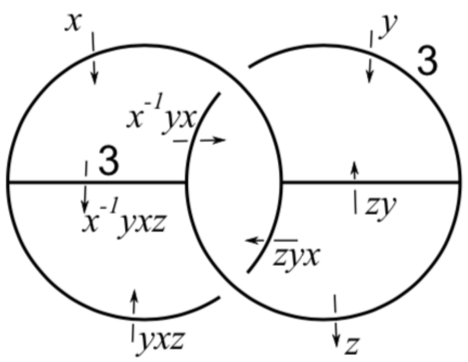


Figure 2.27: fundamental group computation for an orbifold in Table 3 [21]

CHAPTER 3

EXTENDING FINITE GROUP ACTIONS ON SURFACES OVER S^3

As we have already mentioned in the introduction, for a fixed genus g it is hard to determine the maximum order of a finite groups acting on surfaces. So we have a bound for the order, one can ask what is the maximum order of all finite, cyclic or abelian groups that can act on surface of genus g? It has been shown that the $4g+2$ is the bound for cyclic [17] and $4(g+1)$ is the bound for abelian. [8]. But in this chapter, we will be interested in which of these actions can be extendable. First let us recall the Hurwitz theorem.

Theorem 3.0.1. *(Hurwitz,1893[7]) Let Σ_g be a compact surface of genus $g \geq 2$. If G is a finite group acting on Σ_g, then $|G| \leq 84(g-1)$.*

Proof. (Sketch of Proof.) Since the order of the group is finite, the quotient of the action is a compact two-dimensional orbifold $\mathcal{O}$ with the underlying space $X_{\mathcal{O}} = \Sigma_g/G$. The quotient map $q : \Sigma_g \to X_{\mathcal{O}}$ is an orbifold covering map of degree $|G|$. By the Riemann-Hurwitz formula $|G|\chi(X_{\mathcal{O}}) = \chi(\Sigma_g) = 2-2g$. Suppose the underlying space of the orbifold $X_{\mathcal{O}}$ has genus h and N branch points of order $m_1, m_2, \cdots, m_N \geq 2$. By 2.7.2 the Euler characteristic of $\mathcal{O}$ is equal to

$$\chi(\mathcal{O}) = \chi(X_{\mathcal{O}}) - \sum_{i}^{N}(1-\frac{1}{m_i})$$

So, we need to find the values of h, N and m_i's such that $\chi(\mathcal{O})$ is the maximum number possible.

If $h \geq 2$, then $\chi(\mathcal{O}) \leq -2$. If $h = 1$, there must be at least one branch point of order $m_j \geq 2$, which then gives us $\chi(\mathcal{O}) \leq 2-2-(1-\frac{1}{2}) \leq -\frac{1}{2}$. Now, suppose $\chi(X_{\mathcal{O}})$

has genus $h = 0$. If $N \geq 5$, then $\chi(\mathcal{O}) \leq 2 - \sum_i^5 (1 - \frac{1}{m_i}) \leq 2 - \sum_i^5 (1 - \frac{1}{2}) \leq -\frac{1}{2}$. If $N = 4$, then if all $m_j = 2$ then $\chi(\mathcal{O}) = 0$, which can hold only if $g = 1$. So the highest possible $\chi(\mathcal{O})$ can happen if $m_j = 2$ for 3 branch points and for one of them we have $m_j = 3$ then $\chi(\mathcal{O}) = 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - (1 - \frac{1}{3}) = -\frac{1}{6}$. If $N = 4$, then again by considering all possible cases on the orders of m_1, m_2, m_3 we find that the highest possible $\chi(\mathcal{O}) = 2 - \frac{1}{2} - (1 - \frac{1}{3}) - (1 - \frac{1}{7}) = -\frac{1}{42}$. If $N \leq 2$, then $\chi(\mathcal{O}) \geq 1$. Therefore, $\chi(\mathcal{O}) \leq -\frac{1}{42}$ for $g \geq 2$ and $|G|(-\frac{1}{42}) \geq 2 - 2g$ and so $|G| \leq 84(g-1)$.

□

Definition 3.0.1. *Let G be a finite group acting on a genus g surface Σ_g and let Σ_g be embedded in S^3 through $e : \Sigma_g \hookrightarrow S^3$. If G acts also on S^3 so that the restriction of that action to Σ_g is exactly the action of G on Σ_g, then the action on the surface Σ_g is called extendable (through the embedding e over S^3), i.e. $S^3/G|_{\Sigma_g} = \Sigma_g/G$ [20].*

Definition 3.0.2. *Let the action of G be extendable over S^3. An embedding $e_0 : \Sigma_g \hookrightarrow S^3$ is called an unknotted embedding if $S^3 \setminus \Sigma_g$ is a union of two handlebodies.*

This means that S^3 has a Heegaard splitting, $S^3 = H_1 \cup_{\Sigma_g} H_2$ where two handlebodies H_1, H_2 are glued along the surface Σ_g and H_i's are invariant under the action of G.

3.1 Maximum order of groups with extendable actions

Theorem 3.1.1. *Assume that the action of G on the surface Σ_g can be extended to S^3 through some unknotted embedding $e_0 : \Sigma_g \hookrightarrow S^3$ and also assume that $|G| = 12(g-1)$. Then the genus g is one of the following:*

$$g \in \{2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601\} \text{ [20].}$$

Proof. Let H_1, H_2 be the Heegaard splitting of S^3 under the unknotted embedding e_0 such that the action of G on the handlebodies H_1 & H_2 is invariant. Consider the handlebody orbifolds H_i/G. Since a handlebody orbifold is obtained by attaching 3-discal orbifolds along 2-discal orbifolds [21], the 3-ball B^3 is the underlying

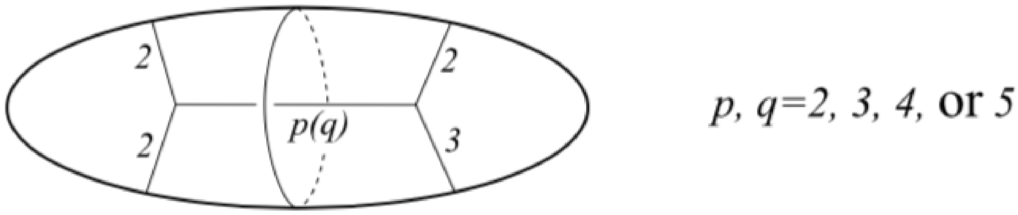


Figure 3.1: singular set of the orbifolds H_i/G [21]

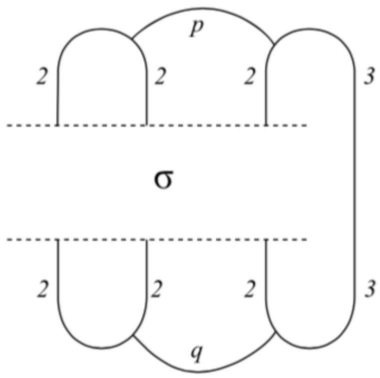


Figure 3.2: singular set of the orbifold S^3/G [21]

space of H_i/G. Moreover, the singular sets of 3-discal orbifolds are classified in Theorem 2.7.3, therefore the singular sets of the handlebody orbifolds are as in the Figure 3.1. Hence, the singular set of the quotient orbifold is as in Figure 3.2, i.e the singular set of the handlebodies are connected to each other according to the branching orders of the strands. The ones with order 3 must be connected to each other to form the singular set of the quotient orbifold. The remaining three strands with branching order 2 can be connected to each other in the shape of a braid on 3 strings, say σ. The quotient orbifold with the resulting singular set is called $O(\sigma; p, q)$ where $p, q = 2, 3, 4, 5$.

The underlying topological space of the quotient orbifold S^3/G is again S^3. Therefore, we look for the orbifolds $O(\sigma; p, q)$ which are spherical. In Dunbar's list [4], all the spherical 3-orbifolds are classified. The orbifolds $O(\sigma; p, q)$ can be divided into two groups; if both p and q are equal to 2 or not. Let us first consider the case where not both p and q are equal to 2.

Case 1 : There must be a non-dihedral singular point (i.e. not of the form $(2, 2, n)$)

in the singular set of the orbifold. So the singularity can be either tetrahedral, octahedral or icosahedral, i.e. A_4, S_4 or A_5. Since the orbifold has underlying space S^2 as it can be seen in Figure 3.1, we are looking for the non-fibered spherical orbifolds which contains the singularities $(2,2,2,3)$. Furthermore, in the singular set of the nonfibered spherical orbifold, the edges with singularities $(2,2)$ and $(2,3)$ intersect at a vertex and those vertices should be connected to each other with another edge. In Dunbar's List of nonfibered spherical orbifolds (see 2.8.2), the singularities of the Figures 2.26h, 2.26i, 2.26m, 2.26n, 2.26o, 2.26q, 2.26r are not of the type $(2,2,2,3)$. Even though the Figure 2.26a has the singularity type $(2,2,2,3)$, there is no edge connecting the vertices with adjacent singularity type $(2,2)$ and $(2,3)$. We also observe that the Figure 2.26c does not have any vertices with adjacent edges of singularity type $(2,2)$ and $(2,3)$. Hence, there are only 9 possible graphs left. (See Figure 3.3)

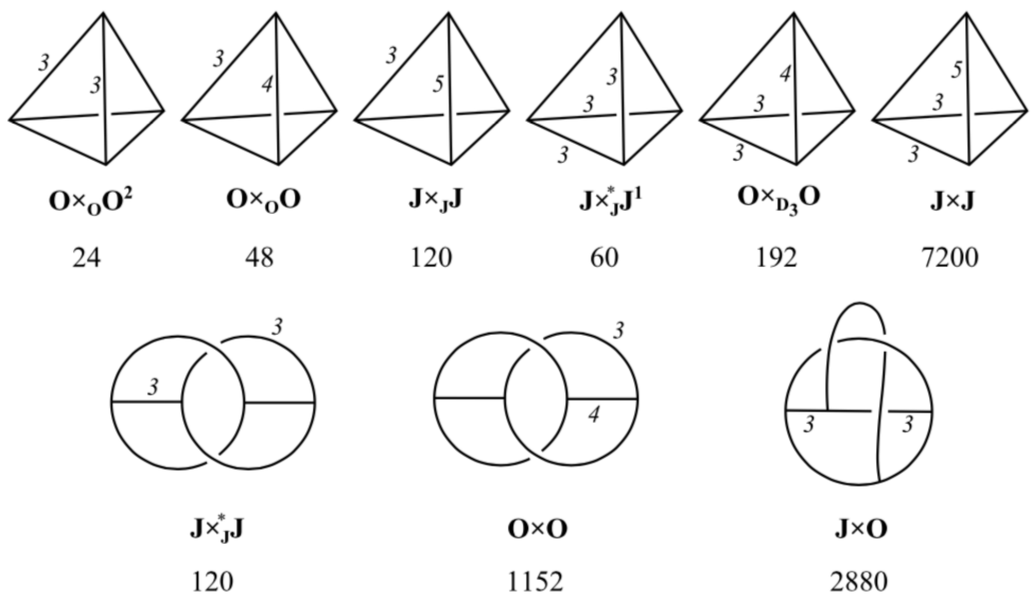


Figure 3.3: nonfibered spherical orbifolds with a singularity type $(2,2,2,3)$ [20]

In [5], Dunbar gives the fundamental group of each of these orbifolds. Note that O is the orientation-preserving symmetry group of octahedron and J is the orientation-preserving symmetry group of icosahedron. So the group G acting on S^3 giving the resulting orbifold is the fundamental group of the orbifold. So, $|G|$ can be 24, 48, 120, 60, 192, 7200, 1152, 2880. Since the relationship between the genus and the group order is $|G| = 12(g-1)$, the genus g can be $3, 5, 13, 6, 17, 601, 97, 241$.

Case 2 : In this case both p and q are equal to 2, i.e. the orbifold is $O(\sigma; 2, 2)$. This quotient orbifold can be obtained in the following way: Consider a two-bridge link $L(\sigma)$ in S^3 and assign the singular index 3 to each components of that link. This represents the singular set of an orbifold, call $L_3(\sigma)$. In Theorem 1 of [10], it is shown that the orbifold $L_3(\sigma)$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ covering of the quotient orbifold $O(\sigma; 2, 2)$. Therefore, if $O(\sigma; 2, 2)$ is a spherical orbifold, then so is the orbifold $L_3(\sigma)$ whose singular set is a two-bridge link. Among all the spherical orbifolds in Section 2.8 ([4]), only the Table 1 of Figure 2.24 consists of the spherical orbifolds with the singular set as a two-bridge link. Those links which have branching index 3 on each of their components are given in Figure 3.4

[20]

Figure 3.4: spherical orbifolds with a two-bridge link singular set [20]

Consider the 3-fold cyclic branched covering of $L_3(\sigma)$ with no singular set, say $\tilde{L_3}(\sigma)$. Then the corresponding orbifold covering map is given by

$$\tilde{L_3}(\sigma) \to L_3(\sigma) \to O(\sigma; 2, 2).$$

Since $L_3(\sigma)/(\mathbb{Z}_2 \times \mathbb{Z}_2) = O(\sigma; 2, 2)$ and $\tilde{L_3}(\sigma) \to L_3(\sigma)$ is a 3-fold cover, the degree of the orbifold covering map is 12. The resulting manifolds $\tilde{L_3}(\sigma)$ are Seifert fiber spaces:

- The first manifold, being the 3-fold cyclic branched covering of S^3 over unknot, is S^3. Note that $\pi_1(S^3) = 1$.
- The 3-fold cyclic branched covering of S^3 over the Hopf link is the Lens space $L(3, 1)$ with $\pi_1(L(3, 1)) = \mathbb{Z}_3$.

- The 3-fold cyclic branched covering of S^3 over the trefoil is given as the quarternion manifold $\mathcal{Q}^3$ where its fundamental group is given as $\pi_1(\mathcal{Q}^3) =< a, b \mid a^2 = b^2 = (ab)^2 >$ (see p.304-305 of [14])

- The manifold obtained by taking the cyclic branched cover over the fourth link has fundamental group $\pi_1 =< x, u \mid x^3 = u^2 = (ux)^2 >$ which has order 24. [14]

- The last singular set is a torus knot of type $(2, 5)$. (see Figure 2.5) By Theorem 1 in p. 309 of [14], the 3-fold cyclic branched cover of S^3 over the torus knot of type $(2, 5)$ is the Poincaré homology 3-sphere, denoted by $\mathcal{P}^3$. Also the fundamental group of $\mathcal{P}^3$ is the binary icosahedral group of order 120.

So, the order of the fundamental groups of $\tilde{L_3}(\sigma)$ are $1, 3, 8, 24, 120$. Since the degree of the orbifold covering is 12, the order of the groups $|G|$ corresponding to the orbifolds $O(\sigma; 2, 2)$ are $12, 36, 96, 288, 1440$. From, $|G| = 12(g-1)$, the possible genera g for an extendable unknotted action are $2, 4, 9, 25, 121$ [20].

$\square$

Lemma 3.1.1. *[20] Suppose a group G is acting on the pair (F, M) where F is a surface embedded in a 3-manifold M with the embedding $i : F \hookrightarrow M$. Then we have the following diagrams:*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ F/G \arrow[r, "\hat{i}"'] & M/G \end{tikzcd} \qquad \begin{tikzcd} \pi_1(F) \arrow[r, "i_*"] \arrow[d, "p_*"'] & \pi_1(M) \arrow[d, "p_*"] \\ \pi_1(F/G) \arrow[r, "\hat{i}_*"'] & \pi_1(M/G) \end{tikzcd}$$

Assume the orbifold F/G is connected. Then F is connected if and only if $\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M)) = \pi_1(M/G)$.

Proof. ($\Leftarrow$) Let $\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M)) = \pi_1(M/G)$ hold and assume that F is not connected. Then F has a connected component F_1 such that $F_1 \subsetneq F$. Let $G_1 < G$

be the stabilizer of F_1, $G_1 = \{h \in G \mid h(F_1) = F_1\}$. Then $F_1/G_1 = F/G$. Observe that $|\pi_1(M/G) : p_*(\pi_1(M))| = |G|$ since $p : M \to M/G$ is a universal cover with G as its deck transformation group. Then

$$|G| = |\pi_1(M/G) : p_*(\pi_1(M))| = |(\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M))) : p_*(\pi_1(M))|$$

by assumption, which is equal to $|\hat{i}_*(\pi_1(F/G))/(\hat{i}_*(\pi_1(F/G)) \cap p_*(\pi_1(M)))|$ by the Second Isomorphism Theorem. Since $\hat{i}_*(\pi_1(F/G) \cap p_*(\pi_1(M))$ is a subgroup of $i_*p_*(\pi_1(F_1))$,

$$|\hat{i}_*(\pi_1(F/G))/(\hat{i}_*(\pi_1(F/G)) \cap p_*(\pi_1(M)))| \leq |\hat{i}_*(\pi_1(F/G)) : \hat{i}_*p_*(\pi_1(F_1))|$$

Since $\hat{i}_*p_*(\pi_1(F_1)) = p_*(\pi_1(F_1)).ker(\hat{i}_*)$,

$$\begin{aligned} |\hat{i}_*(\pi_1(F/G)) : \hat{i}_*p_*(\pi_1(F_1))| &= |\pi_1(F/G)/ker(\hat{i}_*) : p_*(\pi_1(F_1)).ker(\hat{i}_*)/ker(\hat{i}_*)| \\ &= |\pi_1(F_1/G_1) : p_*(\pi_1(F_1)).ker(\hat{i}_*)| \text{ ; } F_1/G_1 = F/G \\ &\leq |\pi_1(F_1/G_1) : p_*(\pi_1(F_1))| \text{ ; } \hat{i}_* \text{is an inclusion} \\ &= |G_1| < |G| \text{ ; a contradiction.} \end{aligned}$$

$\Rightarrow$ Assume that $\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M)) \subsetneqq \pi_1(M/G)$. Then for the orbifold M/G, we can find an orbifold covering space $\hat{M}$ which corresponds to
$\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M))$.

$$\begin{tikzcd} F \arrow[rr, "i"] \arrow[dd, "p"] & & M \arrow[dl] \arrow[dd, "p"] \\ & \hat{M} \arrow[dr, "\hat{p}"] & \\ F/G \arrow[ur, dashed] \arrow[rr, "\hat{i}"] & & M/G \end{tikzcd}$$

By the General Lifting Lemma, F/G lifts to $\hat{M}$ since $\hat{i}_*(\pi_1(F/G)) \subset \hat{p}_*(\pi_1(\hat{M}))$. Also, $\hat{M}$ is a union of disjoint copies, therefore F must be disconnected. This is a contradiction, so we have that $\hat{i}_*(\pi_1(F/G)).p_*(\pi_1(M)) = \pi_1(M/G)$ [21].

□

3.2 Maximum orders of abelian and cyclic groups with extendable actions

Assume that the G-action on the surface Σ_g is extendable to S^3 through some embedding of Σ_g in S^3. Let us define the set
$\Gamma = \{x \in S^3 \mid \exists g \in G \setminus \{id\}$ such that $gx = x\}$. Note that Γ is a graph that might be disconnected. Also, by Proposition 2.7.1 S^3/G gives a 3-orbifold with branch set Γ/G, which is also a graph. Then edges of Γ/G are assigned a positive integer, which is the branch index of the corresponding edge.

Now, suppose that there is a G-action on a handlebody V_g of genus g. The orbit space of this action gives a handlebody orbifold V_g/G. For this handlebody orbifold, a corresponding finite **graph of groups** $(\Gamma, \mathcal{G})$ is determined in the following way:
Γ is a graph where $\mathcal{G}$ assigns a finite group G_v to each vertex of Γ such that $G_v \leq SO(3)$ and assigns a finite group G_e to each edge of Γ such that $G_e \leq SO(2)$. Also if a vertex v is connected to an edge e then there is a monomorphism $G_e \hookrightarrow G_v$. Being associated to V_g/G, there is a map $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$ which is a surjection where its kernel is a free group of rank g. Furthermore, ϕ is injective on each vertex group G_v. The fundamental group $\pi_1(\Gamma, \mathcal{G})$ is obtained by the free product with amalgamation and HNN-extension of the vertex groups of $(\Gamma, \mathcal{G})$ over its edge groups [20].

Note that cyclic, dihedral, tetrahedral, octahedral and icosahedral groups are the only finite subgroups of the orthogonal group $SO(3)$, which correspond to the orbifolds in Theorem 2.7.3. Then the vertex groups G_v are among these five groups. Moreover, since the edge groups G_e are finite subgroups of $SO(2)$ and since they are proper subgroups of their adjacent vertex groups, the edge groups are either trivial group or a cyclic group which is maximal in their adjacent vertex groups. Conversely, if a graph of groups is given corresponding to a handlebody orbifold together with a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$, then corresponding to this surjection, there is a G-action on a handlebody V_g of genus g.

Proposition 3.2.1. *The fundamental group of* $\Gamma(A, C, B)$ *is*
$\pi_1(\Gamma(A, C, B)) = A *_C B$

Definition 3.2.1. *Let* $\chi(\Gamma, \mathcal{G})$ *denote the Euler characteristic of the graph of groups* $(\Gamma, \mathcal{G})$*. Then it is given by* $\chi(\Gamma, \mathcal{G}) := \sum 1/|G_v| - \sum 1/|G_e|$ *where* G_v, G_e *are vertex* *and edge groups respectively.*

Proposition 3.2.2. *Let a finite group G act on a genus $g \geq 2$ handlebody. Then if $\chi = \chi(\Gamma, \mathcal{G})$ is the Euler characteristic of finite graph of groups of the corresponding handlebody orbifold, then $g - 1 = |G|(-\chi)$*

One can find a proof of Proposition 3.2.2 in [23].

Definition 3.2.2. *Let G act on the handlebody V_g. Then any two actions of G is said to be equivalent if the homeomorphism groups of V_g on those actions of G are conjugate.*

Theorem 3.2.1. *[20] Assume that a finite abelian group G acts on a handlebody V_g where $g \geq 2$. Then the maximum order of G is given by $2g + 2$ if $g \neq 5$ and 16 if $g = 5$. Also, for $g \neq 3, 5$ those groups G with maximum orders are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$, while for $g = 3, 5$, $G \cong (\mathbb{Z}_2)^3, (\mathbb{Z}_2)^4$ respectively.*

(i) There is only one equivalence class for every group of the form $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ and also for $(\mathbb{Z}_2)^4$. however, for the group $(\mathbb{Z}_2)^3$, which acts on the handlebody V_3 has three equivalence classes.

(ii) The group $(\mathbb{Z}_2)^4$ is the only abelian group acting on V_5 with order greater than 12.

Proof. By Proposition 3.2.2, as $g \geq 2$, $\chi < 0$. Consider the graph of groups $(\Gamma, \mathcal{G})$. Then its vertex groups G_v are the subgroups of $SO(3)$, which are precisely the cyclic group $\mathbb{Z}_n$, the dihedral group D_{2n}, the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5. Therefore G_v is either $\mathbb{Z}_n$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_4$ since they are the only abelian subgroups of $SO(3)$. Also since the edge groups are finite subgroups of $SO(2)$, they are cyclic and moreover, they are proper in each adjacent vertex group because there is a monomorphism $G_e \hookrightarrow G_v$ between each adjacent edge and vertex groups. If an edge e is not a loop then the corresponding group G_e can be either trivial group or $\mathbb{Z}_2$. In the case where $G_e = \mathbb{Z}_2$, each adjacent vertex group must be $\mathbb{Z}_2 \times \mathbb{Z}_2$ Now, consider the abelian groups with $|G| \geq 2g - 1$. Then one has $-\chi = \frac{g-1}{|G|} \leq \frac{g-1}{2g-1} < 1/2$. Therefore, we will consider the possible graphs $(\Gamma, \mathcal{G})$

(and the corresponding group G) under the assumption that $-\chi < 1/2$. We will examine two different possibilities regarding the vertices with vertex group $\mathbb{Z}_2 \times \mathbb{Z}_2$: Either some of the vertex groups of the graph $(\Gamma, \mathcal{G})$ are $\mathbb{Z}_2 \times \mathbb{Z}_2$ or none of the vertex groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 1: Assume that none of the vertex group of $(\Gamma, \mathcal{G})$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This means for every vertex v, $G_v = \mathbb{Z}_2$. Also define the set $E = \{e \in \Gamma \mid G_e \neq 1\}$. As none of the vertex groups can be $\mathbb{Z}_2 \times \mathbb{Z}_2$, the set E contains only the edges which are actually loops. Denote the corresponding graph having no loops with $\Gamma_0 = \Gamma \setminus E$, and the usual graph having no edge and vertex group with $|\Gamma_0|$. Note that $\chi(\Gamma) = \chi(\Gamma_0) - \sum_{e \in E} 1/G_e$, which implies $\chi(\Gamma) \leq \chi(\Gamma_0)$. Also by using the definition of $\chi(\Gamma_0, \mathcal{G})$ and $\chi(|\Gamma_0|) = \#\ of\ vertices - \#\ of\ edges$, one can see that $\chi(\Gamma_0) \leq \chi(|\Gamma_0|)$. In short, the following inequality holds: $-1/2 < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)$. Moreover, observe that the Euler characteristic of the graph $|\Gamma_0|$ can be either 0 or 1. This is because while attaching an edge, either a new vertex is introduced to the graph or it is attached to two vertices which are present in the graph and forms a cycle.

Let k be the number of nontrivial vertices in the graph $|\Gamma_0|$. Then

$$\chi(\Gamma_0) = \sum_{trivial\ edges} \frac{1}{|G_v|} + \sum_{nontrivial\ edges} \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

$$\leq \sum_{trivial\ edges} \frac{1}{|G_v|} - \sum \frac{1}{|G_e|} + \frac{k}{2} \leq \chi(|\Gamma_0|),$$

or equivalently $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - k/2$.

Now, if $\chi(|\Gamma_0|) = 0$ then $\chi(\Gamma) \notin (-1/2, 0)$, which is a contradiction. Therefore, the second possibility applies: $\chi(|\Gamma_0|) = 1$ and $|\Gamma_0|$ is a tree. Consider the degree one vertices of the graph, which are precisely the ends of the graph. The vertex groups of those ends cannot be trivial, because if they were, there wouldn't be an edge attached to that vertex, so it would be isolated. Therefore, each end of $|\Gamma_0|$ is nontrivial.

Thus the graph Γ is only a ray, consisting of one edge and two vertices with the corresponding groups $1, \mathbb{Z}_{n_1}, \mathbb{Z}_{n_2}$ respectively. Denote it as $\Gamma(\mathbb{Z}_{n_1}, 1, \mathbb{Z}_{n_2})$. Moreover, because the order of the vertex groups are actually the orders of the branch set of the orbifold, the tuple (n_1, n_2) can be only the followings: $\{(2, n), (3, 3), (3, 4), (3, 5)\}$, where $n = 3, 4$ or 5. Hence, one can conclude that the graph Γ has no loops at all, i.e. $\Gamma = \Gamma_0$. This is because a loop occurs only if $(n_1, n_2) = (2, 4)$ so that the edge group is nontrivial. However, in that case the vertex group is $\mathbb{Z}_2 \times \mathbb{Z}_2$, but it is assumed that none of the vertex groups are $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In short, Γ is either $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$ with $\chi = \frac{n-2}{2n}$ $(n \leq 5)$ or $\Gamma(\mathbb{Z}_3, 1, \mathbb{Z}_n)$ where $n = 3, 4, 5$ with $-\chi = 1/3,\ 5/12,\ 7/15$.

Note that the fundamental group of $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$ is $\pi_1(\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)) \cong \mathbb{Z}_2 * \mathbb{Z}_n$.

Also, $\mathbb{Z}_2 \times \mathbb{Z}_n$ and $\mathbb{Z}_n$, (n is even) are the only finite abelian groups surjecting to the free product $\pi_1(\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)) \cong \mathbb{Z}_2 * \mathbb{Z}_n$ such that the kernel is torsion-free. In the first possibility, where Γ is $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$, $g = n - 1$ and $|G| = 2n = 2(g + 1)$. Therefore, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{g+1}$. Now, for the other cases, i.e. Γ is $\Gamma(\mathbb{Z}_3, 1, \mathbb{Z}_n)$ for $n = 3, 4, 5$ the possible group G are $\mathbb{Z}_3$, $(\mathbb{Z}_3)^2$, $\mathbb{Z}_{12}$ and $\mathbb{Z}_{15}$, where $|G| < 2(g + 1)$ in each case.

Case 2: Assume that some of the vertex groups G_v in the graph $(\Gamma, \mathcal{G})$ are $\mathbb{Z}_2 \times \mathbb{Z}_2$. Similar to the first case, let us define the set
$E = \{e \in \Gamma |\ adjacent\ vertex\ groups\ are\ cyclic\ and\ G_e \neq 1\}$. Note that every nontrivial edge with cyclic adjacent vertices has to be a loop. This is because the edge group is a subgroup of each cyclic vertex group, which are $\mathbb{Z}_n$'s with $n = 2, 3, 5$. This is a contradiction since in that case the edge group becomes trivial, therefore there is only one vertex group, i.e, the edge is actually a loop.

So, by letting $\Gamma_0 = \Gamma \setminus E$ we guarantee that every edge with a nontrivial group has two vertices with $G_v = \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, the inequality in the previous case still holds: $-1/2 < \chi(\Gamma) \leq \chi(\Gamma_0) \leq \chi(|\Gamma_0|)$.

Let us look at the last inequality $\chi(\Gamma_0) \leq \chi(|\Gamma_0|)$, because the proof of the rest of the inequality is the duplicate of the one in the first case. Let l be the number of vertices in the graph Γ_0 with the vertex groups $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then since there are at most three edges coming out of a vertex and since any two vertices together define at most one edge, the number of nontrivial edge groups is less than $\frac{3l}{2}$.

Now, $\chi(\Gamma_0)$ has l vertices of order 4 and at most $\frac{3l}{2}$ edges of order 2. We will use the definition of the Euler characteristic of a usual graph and the definition of $\chi(\Gamma_0)$.

As $|\Gamma_0|$ is a usual graph, either $\chi(|\Gamma_0|) = 0$ or $\chi(|\Gamma_0|) = 1$.

Note that a cyclic end $\chi(\Gamma_0)$ is less than $\chi(|\Gamma_0|)$ by at least $1/2$ since a cyclic end decreases the Euler characteristic of the graph of group by at least $1 - \frac{1}{n} = \frac{n-1}{n}$, which is $\leq 1/2$ if $n \geq 2$.

Since there are l vertices of type $\mathbb{Z}_2 \times \mathbb{Z}_2$, there are at most $\frac{3l-2}{2}$ nontrivial edges. Then $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3l}{4} + \frac{1}{2}\frac{3l-2}{2} = \chi(|\Gamma_0|) - \frac{1}{2}$. Therefore, the graph $\chi(|\Gamma_0|)$ can have at most 2 ends.

- If $\chi(|\Gamma_0|) = 0$, then number of vertices is equal to the number of edges. Therefore, Γ_0 is a loop containing some $\mathbb{Z}_2 \times \mathbb{Z}_2$ vertices. So, the graph of groups with only one vertex and one edge is the possibility for $(\Gamma, \mathcal{G})$ with $\chi = -1/4$. However, the $\mathbb{Z}_2$ subgroups of the vertex group $\mathbb{Z}_2 \times \mathbb{Z}_2$ are conjugate to each other by the HNN generator of $\pi_1(\Gamma, \mathcal{G})$ of the loop [16]. Therefore, $\pi_1(\Gamma, \mathcal{G})$ does not surject onto an abelian group. Hence, this case is not possible.

- If $\chi(|\Gamma_0|) = 1$, then Γ_0 is a segment, with possibly inner vertices with $\mathbb{Z}_2 \times \mathbb{Z}_2$. For every inner vertex, there are at most $\frac{3l-1}{2}$ edges with a nontrivial group, thus $\chi(\Gamma_0) \leq \chi(|\Gamma_0|) - \frac{3l}{4} + \frac{1}{2}\frac{3l-1}{2} = \chi(|\Gamma_0|) - \frac{1}{4}$. Therefore, the number of inner vertices is at most one.

  If the segment has no inner vertex, then $(\Gamma, \mathcal{G})$ is either $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2, 1, \mathbb{Z}_2)$ with $-\chi = 1/4$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2, 1, \mathbb{Z}_3)$ with $-\chi = 5/12$. In the first case of $(\Gamma, \mathcal{G})$, the possible groups G are $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $(\mathbb{Z}_2)^3$. In the second case, G is $\mathbb{Z}_2 \times \mathbb{Z}_6$. Note that the upper bound $|G| = 2g + 2$ is achieved only for the group $(\mathbb{Z}_2)^3$ where $g = 3$.

  If there is an inner vertex, the graph of groups is $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$ with $-\chi = 1/4$. Also there is a surjection from its fundamental group onto $(\mathbb{Z}_2)^n$ where $n = 2, 3, 4$. Therefore, the order of the group $(\mathbb{Z}_2)^4$ is maximum with $16 = 2(g + 1)$ for $g = 5$ as in the theorem. Also, the order of $(\mathbb{Z}_2)^3$ is maximum with $8 = 2(g + 1)$ for $g = 3$ again as in the previous case.

Finally, there is one equivalence class for the group $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ since there is only one finite bijection from $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$ to an abelian group.

The possible finite bijections from $\Gamma((\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_2, (\mathbb{Z}_2)^2)$ to $(\mathbb{Z}_2)^3$ are given by either all three vertices are mapped to different subgroups of $(\mathbb{Z}_2)^3$ and given by two vertices are mapped to the same subgroup of $(\mathbb{Z}_2)^3$.Also, there is one finite bijection from $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2, 1, \mathbb{Z}_2)$ to $(\mathbb{Z}_2)^3$. Therefore, in total there are three equivalence classes for $(\mathbb{Z}_2)^3$ acting on V_3. This proves (i).

Note that the abelian groups of order $13, 14, 15$ are cyclic groups. We also know that each of the graph of groups represent the singular set of the corresponding orbifold. There is no finite injective surjection to these cyclic groups for $g = 5$.

□

For a cyclic group G, the previous theorem implies the following:

Theorem 3.2.2. *[20] Assume that G is a finite cyclic group which acts on a handlebody with genus $g \geq 2$. If $|G| \geq 2g - 2$ the G is listed in one of the following groups:*

(i) $\mathbb{Z}_{2g+2}$ *if g is even with a corresponding surjection* $\mathbb{Z}_2 * \mathbb{Z}_{g+1} \to \mathbb{Z}_{2g+2}$

(ii) $\mathbb{Z}_{2g}$*, for every g, with a corresponding surjection* $\mathbb{Z}_2 * \mathbb{Z}_{2g} \to \mathbb{Z}_{2g}$ *and when* $g = 6$*, with the surjection* $\mathbb{Z}_3 * \mathbb{Z}_4 \to \mathbb{Z}_{12}$

(iii) $\mathbb{Z}_{2g-1}$ *for* $g = 2$ *and* $g = 8$ *with surjections* $\mathbb{Z}_3 * \mathbb{Z}_3 \to \mathbb{Z}_3$ *and* $\mathbb{Z}_3 * \mathbb{Z}_5 \to \mathbb{Z}_{15}$

(iv) $\mathbb{Z}_{2g-2}$ *for every g, with the graph of groups* $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$ *with an extra loop attached to a vertex of type* $\mathbb{Z}_n$*, where the edge group is* $\mathbb{Z}_n$*,* $n|2g-2$*; moreover, for* $g = 2$ *and* $g = 3$ *the actions are associated to the surjections* $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2$ *and* $\mathbb{Z}_4 * \mathbb{Z}_4 \to \mathbb{Z}_4$ *respectively.*

Proof. The proof of Theorem 3.2.1 also proves the cases where $|G| \geq 2g - 1$. It is enough to consider the case $|G| = 2g - 2$. Since $g - 1 = |G|(-\chi)$, the Euler characteristic of the possible graphs of groups is $-\chi = 1/2$. Assume that Γ_0 has at least three nontrivial vertices and $g = 2$. Then the fundemental group of $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_2, 1, \mathbb{Z}_2)$ is $\pi_1(\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_2, 1, \mathbb{Z}_2)) = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. There is a surjection from $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ to G with torsion free kernel. Therefore, the associated surjection is $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2$. If $g = 3$, then Γ_0 can be a ray with two vertices, in particular $\Gamma(\mathbb{Z}_4, 1, \mathbb{Z}_4)$ with the surjection

$\mathbb{Z}_4 * \mathbb{Z}_4 \to \mathbb{Z}_4$. Finally, Γ can also be the graph obtained by attaching a loop with $\mathbb{Z}_n$ group to the $\mathbb{Z}_n$ vertex of $\Gamma(\mathbb{Z}_2, 1, \mathbb{Z}_n)$. Then $\chi = 1/2 + 1/n - 1 - 1/n = -1/2$, which gives $|G| = 2g - 2$ [20].

□

Theorem 3.2.3. *Let AE_g be the maximum order of all finite abelian groups G with an extendable action on Σ_g to S^3. Then $AE_g = 2g + 2$.*

Proof. Assume that the action of an abelian group G on the surface Σ_g is extendable to S^3 with some embedding $e : \Sigma_g \hookrightarrow S^3$. The second theorem of [13] proves that an abelian G-action on the surface of Σ_g extends to a compact 3-manifold M such that $\partial M = \Sigma_g$ if and only if it extends to a handlebody V_g such that $\partial V_g = \Sigma_g$. Therefore, $AE_g \leq AH_g$ where AH_g is the maximum order of all finite abelian groups G acting on the handlebody V_g. It is known that if there is a finite orientation preserving group action on S^3 then it is indeed a subgroup of $SO(4)$ by a conjugation. However, $(\mathbb{Z}_2)^4$ is not isomorphic to any subgroup of $SO(4)$. This means that the action of $(\mathbb{Z}_2)^4$ on the surface Σ_5 cannot be extended to S^3. Then the Theorem 3.2.1 implies that for all $g \geq 2$, $AE_g \leq 2g + 2$. Also by Example 4.0.1, the abelian group action $\mathbb{Z}_m \times \mathbb{Z}_n$ on the surface Σ_g is extendable on S^3 with an unknotted embedding $e_0 : \Sigma_g \hookrightarrow S^3$. This implies that $AE_g^o \geq 2g + 2$ where AE_g^o is the maximum order of a finite abelian group which acts on Σ_g extendably to S^3 with some unknotted embedding. Therefore we have $AE_g^o \geq 2g + 2$, which implies: $2g + 2 \leq AE_g^o \leq AE_g \leq 2g + 2$. Hence $AE_g^o = AE_g = 2g + 2$ is proved [20].

□

Theorem 3.2.4. *Let CE_g be the maximum order of all finite cyclic groups G with an extendable action on Σ_g to S^3. Then* $CE_g = \begin{cases} 2g+2 & if\ g\ is\ even \\ 2g-2 & if\ g\ is\ odd \end{cases}$

Proof. Note that in Example 4.0.1 an action of $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ on the surface Σ_g is constructed which extends to S^3 if $g > 1$ is even. Also Example 4.0.2 gives a $\mathbb{Z}_{2g-2}$ action on Σ_g extending to S^3 if $g > 1$ is odd. Therefore, $CE_g \geq 2g + 2$ for even $g > 1$ and $CE_g \geq 2g - 2$ for odd $g > 1$.

Now, assume that the action of G on Σ_g is extendable. The second theorem of [13] implies that the G-action on Σ_g extends to $(V_g, \partial V_g = \Sigma_g)$. By Theorem 3.2.2 part (i), for every even g, $AH_g = 2g+2$ where AH_g is the maximum order of all finite abelian groups acting on the handlebody V_g. The parts (ii) and (iii) of the same theorem implies that for every odd g, a cyclic group G acting on V_g with $|G| > 2g-2$ is $\mathbb{Z}_{2g}$ with an associated surjection $\mathbb{Z}_2 * \mathbb{Z}_{2g} \to \mathbb{Z}_{2g}$.

Claim: The action of $\mathbb{Z}_{2g}$ on $\partial V_g = \Sigma_g$ which extends to V_g is not extendable.

Note that the claim implies $CH_g \leq 2g+2$ for even $g > 1$ and $CH_g \leq 2g-2$ for odd $g > 1$.

Proof of the Claim: Consider the action of $\mathbb{Z}_{2g}$ on V_g. The handlebody orbifold $X = V_g/\mathbb{Z}_{2g}$ consists of two 3-balls with an arc as their singular sets having indices 2 and $2g$, connected by a 1-handle as in Figure 3.5.

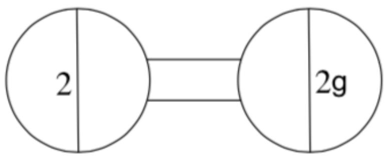


Figure 3.5: the handlebody orbifold $V_g/\mathbb{Z}_{2g}$ [20]

The preimage of the 3- ball with a singular arc of index $2g$ under the map $V_g \to V_g/\mathbb{Z}_{2g}$ is a 3-ball B^3 in V_g. The action of $\mathbb{Z}_{2g}$ on that 3-ball is a $\frac{\pi}{g}$-rotation. The preimage of the remaining part of X in V_g is B^3 where g 1-handles are attached to opposite $\mathbb{Z}_{2g}$-equivariant discs on B^3.

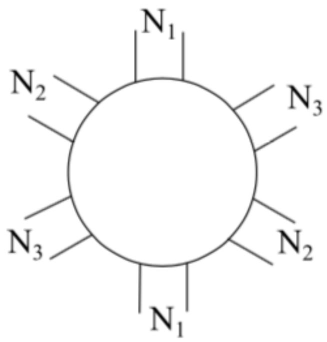


Figure 3.6: preimage of X in V_3 [20]

The Figure 3.6 shows the case for $g = 3$.

Therefore, the $\mathbb{Z}_{2g}$ action on the surface $\Sigma_g = \partial V_g = S^2_* \cup \{N_1, ..., N_g\}$ is obtained from a punctured 2-sphere S^2_* by $2g$ punctures where g tubes $N_1, ..., N_g$ along g-many opposite puncture pairs. We will know prove that the given $\mathbb{Z}_{2g}$ action on Σ_g is not extendable.

Let γ be an arc on ∂X as in the Figure 3.7. The preimage of γ under the orbit map of X contains g arcs γ_i, $i = 1, ..., g$ on $\partial V_g = \Sigma_g$, which are equivariant under the G-action. Let us denote the upper hemisphere of S^2_* with $2g$ punctures by D. Then the boundaries of the arcs γ_i divide the boundary of D into $2g$ arcs, denoted by α_i and β_i such that α_i is mapped to β_i under the action of $\frac{\pi}{g}$-rotation. (Figure 3.8)

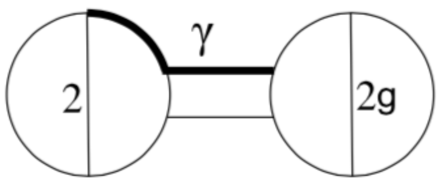


Figure 3.7: the arc γ on ∂X [20]

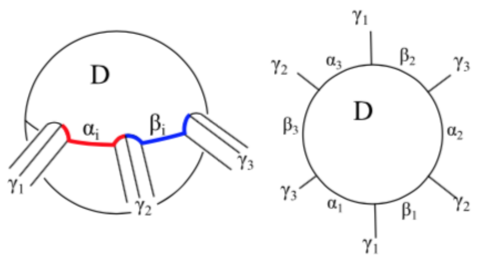


Figure 3.8: α_i and β_i on D [20]

Now, let $K_1 = \gamma_i \cup \alpha_i$ and $K_2 = \gamma_i \cup \beta_i$. Since the surface Σ_g is embedded into S^3, K_1 and K_2 are knots in S^3 as in the Figure 3.9

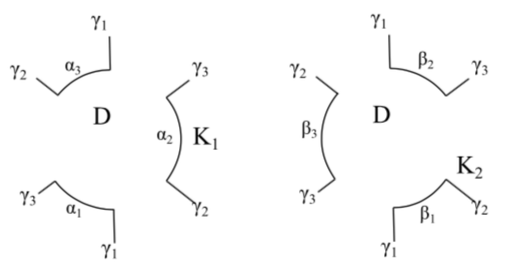


Figure 3.9: the knots K_1 and K_2 in S^3 [20]

Assume that the action of G extends to S^3. Also, let $\sigma \in G$ be the generator of the group G so that $\sigma|_D$ is $\frac{\pi}{g}$-rotation. Then $\sigma(K_1) = K_2$. The fixed point set of σ is nonempty, $\sigma(x) = x$ for some $x \in D$. Let K_0 be the circle component of the fixed point set of σ, passing through x. Then, $\sigma\{K_0, K_1\} = \{K_0, K_2\}$.

Let $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$ be the mod2 linking numbers of K_0, K_1 and K_0, K_2. Let S^2 be a sphere with a standard embedding into S^3, which also contains the disc D. Consider the projections of the knots K_i and K_0 onto this 2-sphere. We will compute the linking numbers $lk_2(K_0, K_i)$ from these projected diagrams. For every crossing, if the arc of K_0 goes over the arc of K_i, then the contribution of this crossing to the linking number $lk_2(K_0, K_i)$ is 1. On the other hand, if the arc of K_0 goes under the arc of K_i, then the contribution of this crossing to the linking number $lk_2(K_0, K_i)$ is 0.

Note that $lk_2(K_0, K_1)$ and $lk_2(K_0, K_1)$ differs from each other only at the crossings of K_0 with the boundary of D. There are only three possible cases: (Figure 3.10)

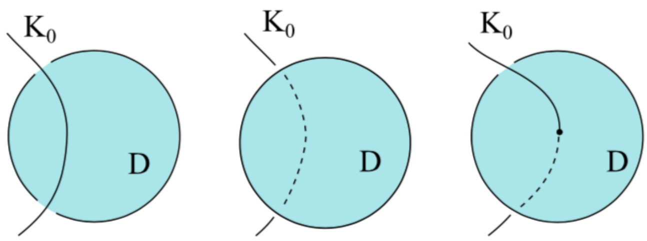


Figure 3.10: three different positions of K_0 and D [20]

Case 1: Both ends of the arc of K_0 in the disc can go over ∂D. There are two possibilities. The two ends of the arc can go over both α_i and β_i. In that case, the contribution of the crossing to the linking numbers $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$ is 0. Also, one of the ends of the arc can go over α_i and the other one can go over β_i then the crossing contributes 1 to the linking numbers $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$.

Case 2: Both ends of the arc of K_0 in the disc can go under ∂D. In that case, the crossing contributes 0 to the both linking numbers $lk_2(K_0, K_1)$ and $lk_2(K_0, K_2)$.

Case 3: One end of the arc in the disc can go over ∂D while the other end goes under ∂D. Then, the arc of K_0 has nonempty intersection with the interior of D, which is the unique fixed point $x \in D$. This corresponds to a unique arc. If the top end of the arc goes over α_i then it contributes 1 to $lk_2(K_0, K_1)$ while it contributes 0 to $lk_2(K_0, K_2)$. On the other hand, if the top end of the arc goes over β_i then it contributes 0 to $lk_2(K_0, K_1)$ while it contributes 1 to $lk_2(K_0, K_2)$. In any case, the contribution cannot be the same for both linking numbers.

So, $lk_2(K_0, K_1) \neq lk_2(K_0, K_2)$. This is a contradiction since $\sigma : S^3 \to S^3$ is an automorphism which maps $\{K_0, K_1\}$ to $\{K_0, K_2\}$. Hence the action of G on the surface Σ_g does not extend to S^3 for $|G| = 2g$.

$\square$

CHAPTER 4

SOME EXAMPLES OF ACTIONS EXTENDING TO S^3

Example 4.0.1. *We will give a $\mathbb{Z}_m \times \mathbb{Z}_n$- action on S^3 by seeing the 3-sphere as a unit sphere lying inside the complex plane $\mathbb{C}^2$:*

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

Consider the points a_j, b_k on S^3 given by $a_j = (e^{\frac{2\pi ji}{m}}, 0)$ for $j = 0, 1, ..., m$ and $b_k = (0, e^{\frac{2\pi ki}{n}})$ for $k = 0, 1, ..., n$. Then a $\mathbb{Z}_m \times \mathbb{Z}_n$ action on S^3 can be generated by the maps $x : (z_1, z_2) \mapsto (e^{\frac{2\pi i}{m}} z_1, z_2)$ and $y : (z_1, z_2) \mapsto (z_1, e^{\frac{2\pi i}{n}} z_2)$. Observe that this is an orientation-preserving action as the maps x and y generating the action represent a rotation in the first and second components. Also, it is a faithful action since there is no nonidentity group element fixing each point (z_1, z_2) in S^3.

Moreover, note that this action generated by x and y sends each a_j to some a_j again (each b_k to some b_k respectively), therefore the set $\{a_j, b_k\}$ is invariant under the $\mathbb{Z}_m \times \mathbb{Z}_n$ action. Then, we can construct a graph by connecting the elements of the set $\{a_j, b_k\}$ with geodesics of S^3, say Γ. This graph Γ is the fixed point set of the action and it has $m + n$ vertices. From each vertex a_j of Γ, there is an edge connecting a_j to each b_k. Therefore, in total, the graph Γ has mn edges, which gives the Euler characteristic of the graph as $\chi(\Gamma) = m + n - mn$. Now, consider the tubular neighborhood of Γ in S^3. This actually forms a handlebody V_g of genus $g = (m - 1)(n - 1)$ since the graph has $(m - 1)(n - 1)$ holes when it is viewed in $\mathbb{R}^3$.

Then, since Γ is sent to itself under $\mathbb{Z}_m \times \mathbb{Z}_n$ action, the action naturally extends to the handlebody V_g also.

(i) Consider the case $m = 2$ and $n = g + 1$. Then the action of the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ on the surface Σ_g extends to S^3 where the embedding of the surface in S^3 is the standard embedding. Furthermore, the action on the surface extending to S^3 is cyclic if g is even, say $g = 2k$, since the group becomes $\mathbb{Z}_2 \times \mathbb{Z}_{2k+1} \cong \mathbb{Z}_{4k+2}$.

(ii) Now we will give an example of a larger group G acting on the handlebody V_g and extending to S^3. Observe that the map $t : (z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$ gives an action of order 2. Similarly, the set $\{a_j, b_k\}$ is invariant under the action of t. Therefore, the group is given as $G \cong \mathbb{Z}_m \times \mathbb{Z}_n \rtimes \mathbb{Z}_2$ acts on the handlebody V_g and its action is extendable over S^3.

A more specific case shows that $G \cong D_{2g+2} \times \mathbb{Z}_2$, a group of order $4(g + 1)$, when $m = 2$ and $n = g + 1$. If $g = 2$, then this action of order 12 is a maximal action.

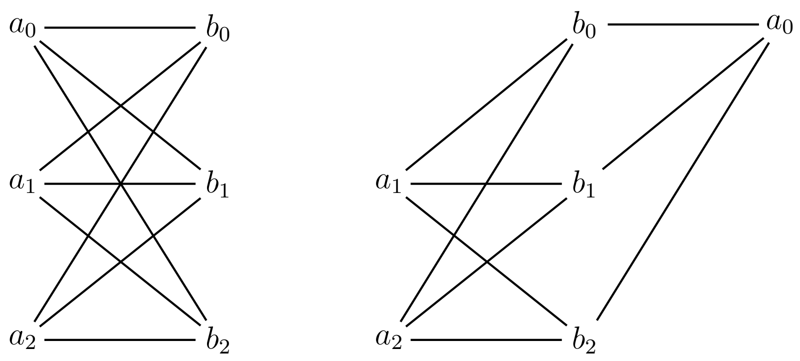
(iii) Define another action $s : (z_1, z_2) \mapsto (z_2, z_1)$ of order 2, where $\{a_j, b_k\}$ is invariant. In this case the group G with extendable action on V_g is given by the semidirect product $G \cong (\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes_\phi (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

For the case where $m = n = k + 1$ for some k , the semidirect product of the groups is given by the following relations:

$$sxs^{-1} = y,\ sys^{-1} = x,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1}$$

where $\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1} =< x, y \mid xy = yx,\ x^{k+1} = y^{k+1} = 1 >$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 =< s, t \mid st = ts,\ s^2 = t^2 = 1 >$.

The graph Γ we constructed before, by joining each a_j to every b_k has $m + n = 2k + 2$ vertices and $mn = (k + 1)^2$ edges. Therefore, $\chi(\Gamma) = 2k + 2 - (k + 1)^2 = -k^2 + 1$. Now, consider a neighborhood of the graph Γ, which forms a handlebody in S^3. Also, the genus of that handlebody is $g = k^2$ as discussed before.



For example, the above graphs represent the same graph, which is for the case $m = n = 3$. However, the second one makes it easier to see that the neighborhood of the graph has indeed genus $g = 4$. In that case, the order of the group with extendable maximal action is $4mn = 36$.

Example 4.0.2. *Let us construct a $\mathbb{Z}_{2g-2}$ action on the handlebody V_g where g is odd. Let S^3 be the unit sphere in $\mathbb{C}^2$, i.e. $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$. Also, note that a solid torus T in $\mathbb{C}^2$ is given by $T = \{(z_1, z_2) \in S^3 \mid |z_1| \leq \frac{\sqrt{2}}{2}\}$.*

Let us choose $g-1$ pairs of points (a_k, b_k), $k = 1, ..., g-1$ such that $a_k = (\frac{\sqrt{2}}{2}e^{\frac{2k\pi i}{g-1}}, \frac{\sqrt{2}}{2}e^{\frac{k\pi}{g-1}})$ and $b_k = (\frac{\sqrt{2}}{2}e^{\frac{2k\pi i}{g-1}}, \frac{\sqrt{2}}{2}e^{\frac{(k+g-1)\pi i}{g-1}})$

Consider the disc D_k containing the points a_k and b_k, i.e. $D_k = \{(re^{\frac{2k\pi i}{g-1}}, z_2) \in S^3 \mid r \geq \dfrac{\sqrt{2}}{2}\}$. Then there is a diameter γ_k of D_k connecting a_k to b_k. Consider the tubular neighborhood N_k of the diameter γ_k in S^3. If one sees the tubular neighborhoods N_k ($k = 1, ..., g-1$) as 1-handles attached to the solid torus T then this gives an embedding of the handlebody V_g into S^3.

Hence, the $\mathbb{Z}_{2g-2}$ action on S^3 is given by $\sigma : S^3 \to S^3$ such that $(z_1, z_2) \mapsto (z_1 e^{\frac{2k\pi i}{g-1}}, z_2 e^{\frac{k\pi i}{g-1}})$. T is invariant under this action and maps N_k to $N_{k+1} mod(g-1)$. (see Figure 4.1 for the action when $g = 5$)

Example 4.0.3. *Let $S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$. Then we can embed S^2 into S^3 by letting $S^2 = \{(x, y, z, 0) \mid x^2 + y^2 + z^2 = 1\}$. Let G be a finite subgroup of $O(3)$ acting on the sphere S^2. We will define an action on S^3 by using the action of G on S^2. For every $\sigma \in G$, define $\tilde{\sigma} : S^3 \to S^3$ by*

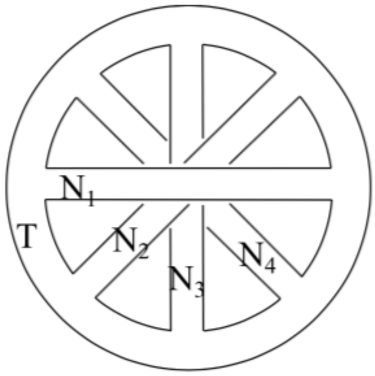


Figure 4.1: $\mathbb{Z}_{2g-2}$ action on $\mathrm{V}_5[20]$

$$\tilde{\sigma}(x,y,z,w) = \begin{cases} (\sigma(x,y,z),w) & if\ \sigma\ is\ orientation-preserving \\ (\sigma(x,y,z),-w) & if\ \sigma\ is\ orientation-reversing \end{cases}$$

Then the group $\tilde{G}$ consisting of the elements $\tilde{\sigma}$ is a finite subgroup of $SO(4)$ and S^2 is invariant under the action of $\tilde{G}$. Now consider a regular tetrahedron with vertices $\{v_1, v_2, v_3, v_4\}$ lying on the sphere S^2 and let G be the symmetry group of the tetrahedron, i.e. $G \cong \mathcal{S}_4$. In that case, the set of vertices is invariant under the action of $G \cong \mathcal{S}_4$ and $\tilde{G}$. Let X be a punctured sphere S^2 from the points $\{v_1, v_2, v_3, v_4\}$. Also, call the tubular neighborhood $N(X)$ of X in S^3 as V_3, which is a handlebody with genus 3. Then the group $\tilde{G}$ acts on the pair (V_3, S^3). Since the map $\phi : G \to \tilde{G}$ defined by $\phi(\sigma) = \tilde{\sigma}$ gives an isomorphism, $|\tilde{G}| = |S_4| = 24$. Since $\tilde{G}$ acts also on the boundary of V_3, which is a surface with genus 3, this case is an example of an extendable group action on a genus $g = 3$ surface with $|\tilde{G}| = 24 = 12(g-1)$.
Similarly, by letting the group G to be the symmetry group of a cube or a dodecahedron, we can find the group $\tilde{G}$ as the groups $S_4 \times \mathbb{Z}_2$ and $A_5 \times \mathbb{Z}_2$ for $g = 5, 11$ respectively. Both of the examples also satisfy the property $|\tilde{G}| = 12(g-1)$.

Example 4.0.4. *Let us denote the 4-dimensional regular Euclidean simplex by Δ, and denote the 4-dimensional Euclidean cube centered at the origin and inscribed in S^3 by Θ. The radial projections of the boundaries of Δ and Θ to S^3 give two regular tessellations of the 3-sphere, denoted by $\partial\Delta$ and $\partial\Theta$. Let v_i be the number of the i-th cells for the tessellations of $\partial\Delta$. Then they are computed as $v_0 = \binom{5}{1} = 5$, $v_3 = \binom{5}{4} = 5$, $v_1 = \binom{5}{2} = 10$ and $v_2 = \binom{5}{3} = 10$. The boundary surface of the regular* *neighborhood of the* 1*-skeleton of this tessellation will give the surface* Σ_g*. Note that the genus of this surface needs to be* $g = v_1 - v_0 + 1$*, in this case* $g = 10 - 5 + 1 = 6$*. Every* 3*-dimensional face of this tessellation is a tetrahedron, whose symmetry is of order* 12*. Since* $v_3 = 5$*, a group* G *of order* $12 \times 5 = 60$ *is acting on the tessellation* $\partial\Delta$ *and on* Σ_6*. Hence,* $60 = 12(g-1)$ *is satisfied for* $g = 6$*, which is an example of an extendable action on* Σ_6 *to* S^3*.*

Now, consider $\partial\Theta$*. It is indeed a product of a cube with an interval. Note that for this tessellation we have* $v_0 = 8 \times 2 = 16$*,* $v_1 = 12 + 12 + 8 = 32$*,* $v_2 = 24$ *and* $v_3 = 8$*. Therefore, the genus of the boundary surface of the regular neighborhood of its* 1*-skeleton is* $g = 32 - 26 + 1 = 17$*. The symmetry group of the every* 3*-dimensional face, which is a cube, has order 24. Since,* $v_3 = 8$*,* $8 \times 24 = 192 = 12(g-1)$ *is the order of the group* G *acting on the surface* Σ_{17} *which extends to* S^3*.*

Example 4.0.5. *The Poincaré homology* 3*-sphere can be obtained by identifying pairs of faces of a dodecahedron. It is also can be seen as the quotient space of the action of the binary icosahedral group* I^* *on* S^3*, i.e.* S^3/I^**. Note that the quotient map* $p : S^3 \to S^3/I^*$ *gives a tessellation for* S^3 *by dodecahedrons. The order of its deck transformations group is* 120 *while the order of the symmetry group of dodecahedron is* 60*. Therefore a group* G *of order* $|G| = 120 \times 60 = 7200$ *is acting on this tessellation of* S^3 *and also on the boundary surface of the regular neighborhood of its* 1*-skeleton, which is* Σ_g*. Let* v_i *be the number of the* i*-th cells of the tessellation. The genus g of the surface is given by* $g = v_1 - v_0 + 1$*. Note that* $\chi(S^3) = v_0 - v_1 + v_2 - v_3 = 0$ *and* $v_3 = 120$*,* $v_2 = 120 \times 12/2 = 720$*. So,* $v_1 - v_0 = v_2 - v_3 = 600$ *and hence* $g = 601$*. This is an example of a maximal extendable group action on the surface* Σ_{601} *with* $|G| = 12(g-1)$*.*

Example 4.0.6. *Let* P *be an oriented pair of pants with an orientation induced on* $\partial P = \{c_1, c_2, c_3\}$ *and let* S^1 *be represented by an oriented curve* h*, as in the Figure 4.2.*

Let $M' = P \times S^1$*. The three boundaries of* M' *are three tori, being* $S^1 \times S^1$*. Glue three solid tori* N_i *to these boundaries so that the meridian of* N_i *is glued to a curve* $l_i = 2c_i + h$*, for* $i = 1, 2, 3$*. Let us denote this manifold by* M*.*

Then M *has the following properties:*

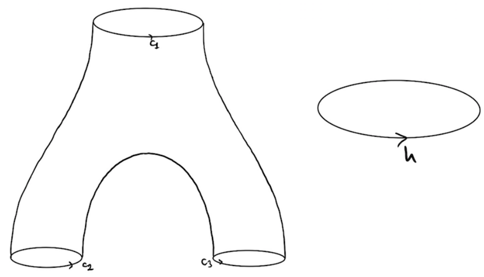


Figure 4.2: an oriented pair of pants and an oriented circle

(i) Note that the symmetry group of of the pair of pants is given by $G_P = D_3 \times \mathbb{Z}_2$, where the action of D_3 on P is orientation-preserving and the action of $\mathbb{Z}_2$ on P is orientation-reversing, which interchanges the inner and outer surface of P. The $G_P = D_3 \times \mathbb{Z}_2$ action on P can be extended to an orientation preserving action on $P \times S^1$ by sending D^3 to the identity of S^1 and and $\mathbb{Z}_2$ to an orientation reversing involution of S^1.

(ii) Since $\pi_1(P) \subseteq \pi_1(P) \times \pi_1(S^1)$ and the map $\pi_1(P) \times \pi_1(S^1) \to \pi_1(M)$ is a surjection, the map $\pi_1(P) \to \pi_1(M)$ is also a surjection.

$$\begin{tikzcd}
\pi_1(\partial(P \times I)) \arrow[dr, two heads] \arrow[ddrr, dotted, bend left, two heads] & & \\
& \pi_1(P) \arrow[d, hook] \arrow[dr, dotted, two heads] & \\
& \pi_1(P) \times \pi_1(S^1) \arrow[r, two heads] & \pi_1(M)
\end{tikzcd}$$

Note that the boundary of the regular neighborhood of P in M is a genus 2 surface Σ_2. To see this, first observe that we can see the pair of pants as a 2-disc with two open discs removed. Then $P \times S^1$ can be seen as a genus 2 surface if S^1 is considered as an interval I identified from its boundaries.

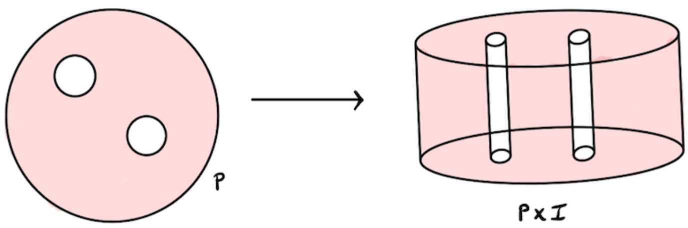


Figure 4.3: obtaining M' from the pair of pants

Also, $\partial(P \times I) \to \pi_1(P)$ is a surjection. From the diagram above, we see that $\pi_1(\partial(P \times I)) \to \pi_1(M)$ is also a surjection. On the other hand, we observed that $\partial(P \times I)$ is Σ_2. Therefore, $\pi_1(\Sigma_2) \to \pi_1(M)$ is a surjection.

(iii) M is a spherical 3-manifold with $\pi_1(M) = \mathbb{Z}_3 \times Q_8$ where Q_8 is the quaternion group, and $|\pi_1(M)| = 24$ [12].

Consider the map $p : S^3 \to M$. By (ii) and Lemma 3.1.1 the preimage of Σ_2 in M is connected. Also, we know that Σ_2 has a covering Σ_h such that $\chi(\Sigma_h) = 2 - 2h = \pi_1(M)\chi(\Sigma_2)$, so $h = 25$. Therefore the surface Σ_{25} is invariant under the action of the group of order $24 \times 12 = 12(g-1)$.

Example 4.0.7. *Consider the link with three components given in the Figure 4.4 and perform -1 surgery on it. By finding its Wirtinger presentation one can find the fundamental group of the resulting manifold M as $\pi_1(M) =< x, y, z \mid x = yz,\ y = zx,\ z = xy >$ (p.305,[14]). If $(x, y, z) \mapsto (i, j, k)$ then $\pi_1(M) \cong Q_8$. So, the manifold M is indeed S^3/Q_8.*

Choose two points in $\mathbb{R}^3$ such that one of them is in front of the link and the other is behind the link. Then connect these two points with strings which pass through every link component, and form the θ-graph shown in the Figure 4.4. The boundary surface of a regular neighborhood of the θ-graph is a genus 2-surface Σ_2. Then there is a group G acting on S^3 keeping Σ_2 invariant and of order 12. The G-action also keeps the link invariant and it extends to the surgered solid tori. The lifted action on S^3 has order 96. Note that $\pi_1(\theta)$ is generated by $x^{-1}z$ and $x^{-1}y$. Therefore, the homomorphism $\pi_1(\theta) \to \pi_1(S^3/Q_8)$ is surjective. Then by the Lemma 3.1.1, the lift of Σ_2 is connected. As in the previous example, we obtain a surface Σ_h such that $2-2h = 8\times(2-2\times 2)$. So, the action is extendable on Σ_9 with $|G| = 12(9-1) = 96$.

Moreover, one can obtain the Poincaré homology 3-sphere P by performing $+1$ surgery on the trefoil knot given in Figure 4.5. Its fundamental group is computed in [14] as $\pi_1(P) =< u, x \mid u^3 = x^5 = (xu)^2 >$ where $u = yx$. It is indeed isomorphic to the binary icosahedral group I^* with $u \mapsto \frac{1+\sigma i+\delta j}{2}$ and $x \mapsto \frac{\sigma+i-\delta k}{2}$ for $\sigma = \frac{\sqrt{5}+1}{2}$ and $\sigma = \frac{\sqrt{5}-1}{2}$.

The σ-graph is constructed similar to the above. The boundary of its regular neigh-

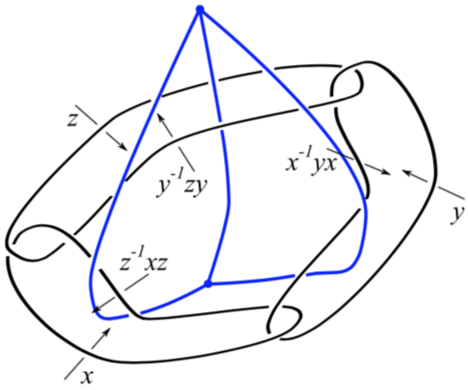


Figure 4.4: θ-graph for the link with three components [20]

borhood will give a surface Σ_2 such that its lift is connected by Lemma 3.1.1, hence is Σ_{121}.. Hence, there is an extentable action of a group G with $|G| = 120 \times 12 = 1440$ on Σ_{121}.

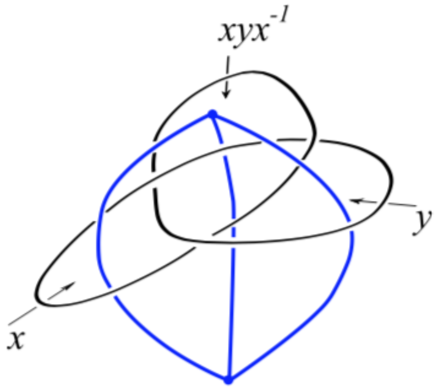


Figure 4.5: θ-graph for trefoil [20]

Example 4.0.8. *Let O denote the octahedral group of order 24. Let $\boldsymbol{O} \times \boldsymbol{O}$ denote the preimage of $O \times O$ under the $2 : 1$ map $SO(4) \to SO(3) \times SO(3)$. Then $\boldsymbol{O} \times \boldsymbol{O}$, of order $24 \times 24 \times 2 = 1153 = 12(97 - 1)$, acts on S^3. It is shown in p.129 of [5] that its prefundamental domain is a truncated cube inscribed in S^3.*

In [5], a prefundamental domain centered at $1 \in S^3$ is defined to be the set of all points in S^3 which are closer to 1 than to any point in the orbit of $1 \in S^3$. After forming the prefundamental domains, fundamental domains in S^3 for the group actions are determined, which gives the rules for gluing the domains resulting in their *quotient spaces.)*

Let two of these truncated cubes be adjacent by an octagon. Then one can get a graph by drawing an edge between the centers of these prefundamental domains. the boundary surface of a regular neighborhood of this graph gives a surface Σ_{97}*, on which* $\boldsymbol{O} \times \boldsymbol{O}$ *has an extendable action.*

Similarly, consider the preimage of $O \times J$ *acting on* S^3*, denote it by* $\boldsymbol{O} \times \boldsymbol{J}$*. Note that it has order* $24 \times 60 \times 2 = 2880 = 12(241 - 1)$*. In [5], its prefundamental domain is given as a twice truncated tetrahedron. If two of these domains are adjacent to each other by a dodecagon, then there is a graph when their centers are connected with an edge. Hence the boundary surface of a regular neighborhood of this graph a surface* Σ_{241}*, on which* $\boldsymbol{O} \times \boldsymbol{J}$ *has an extendable action.*

CHAPTER 5

CONCLUSIONS

In this thesis, the aim has been to give a survey of extendable finite group actions on surfaces to 3-sphere. To reach this aim, we have laid down the motivation in chapter 1 and have given necessary basic background from group actions, low dimensional topology and orbifold theory in chapter 2.

In chapter 3, we have given a detailed proof of the theorem if the action of a finite group G of order $12(g-1)$ on Σ_g can be extended to S^3 through some unknotted embedding, then the genus of the surface is $\{2, 3, 4, 6, 9, 11, 17, 25, 97, 121, 241, 601\}$. The proof was based on the theory of the handlebody orbifolds, since an unknotted embedding gives a Heegaard splitting of S^3. Other two important theorems that we have presented detailed proofs in this chapter were about the maximum orders of the abelian and cyclic groups that have extendable actions on Σ_g to S^3. The maximum orders for abelians it has to be $2g + 2$ and for cyclic ones it is $2g + 2$ if g is even and $2g - 2$ if g is odd. We have given some interesting examples of extendable actions in chapter 4.

One can also wonder about what happens if the embedding is knotted. There has also been some results on that a detailed explanation can be found in [21]. Let us denote the maximal order of the finite group which has an extendable action on Σ_g to S^3 with respect to some embedding by OE_g and OE_g^u if the embedding is unknotted and OE_g^k if the embedding is knotted. We have seen that $4(g+1) \leq OE_g^u \leq OH_g \leq 12(g-1)$ where OH_g be the maximal order of all finite groups which can act on V_g. It is a result due to Zimmermann [9] that $4(g + 1) \leq OH_g \leq 12(g - 1)$, moreover OH_g is either $12(g - 1)$ or $8(g - 1)$ if g is odd. However, in general OH_g are not determined yet.

For an extendable action, if $|G| > 4(g-1)$, all possible relations between the order of G and genus g are listed in the following table. The subindex '$_k$' means the action is realized only for a knotted embedding, the subindex '$_{uk}$' means the action can be realized for both unknotted and knotted embeddings. If the action is realized only for an unknotted embedding, there is no subindex.

$\|G\|$	g
$12(g-1)$	$2, 3, 4, 5, 6, 9_{uk}, 11_{uk}, 17, 25, 97, 121_{uk}, 241_{uk}, 601$
$8(g-1)$	$3, 7, 9, 49, 73$
$20(g-1)/3$	$4, 16, 19, 361_{uk}$
$6(g-1) - I$	$2, 3, 4, 5, 9_{uk}, 11, 17, 25, 97, 121_{uk}, 241_{uk}$
$6(g-1) - II$	$\{2, 3, 4, 5, 9, 11, 25, 97, 121, 241\}_{uk}, 21_k, 481_k$
$24(g-1)/5$	$6, 11, 41, 121$
$30(g-1)/7$	$8, 29, 841, 1681$
$4n(g-1)/(n-2)$	$(n-1), (n-1)^2$

Observe that, except finitely many cases we have $OE_g^u > OE_g^k$ and for finitely many g we have $OE_g^u = OE_g^k$ but we also have $OE_g^u < OE_g^k$ for $g = 21, 481$. There are also some g such that $OE_g^k = 12(g-1)$.

Even though we have these above relations between the genus and the maximum order of extendable actions on Σ_g to S^3, for a fixed genus we do not know which actions are extendable. For future study one can look at the problem of classifying the extendable finite group actions on a fixed low genus g surface.

REFERENCES

[1] Colin C. Adams. *The knot book*. American Mathematical Society, Providence, RI, 2004. An elementary introduction to the mathematical theory of knots, Revised reprint of the 1994 original.

[2] M. A. Armstrong. *Groups and symmetry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988.

[3] Daryl Cooper, Craig D. Hudgson, and Steven P. Kerckhoff. Chapter 2. orbifolds.

[4] William D. Dunbar. Geometric orbifolds. *Rev. Mat. Univ. Complut. Madrid*, 1(1-3):67–99, 1988.

[5] William D. Dunbar. Nonfibering spherical 3-orbifolds. *Trans. Amer. Math. Soc.*, 341(1):121–142, 1994.

[6] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[7] Adolf Hurwitz. Ueber algebraische gebilde mit eindeutigen transformationen in sich. (german). *Math. Ann.*, 41(3):403–442, 1893.

[8] Colin Maclachlan. A bound for the number of automorphisms of a compact riemann surface. *J. London Math. Soc.*, 44:265–272, 1969.

[9] Miller A. McCullough, D. and Bruno Zimmermann. Group actions on handlebodies. *Proc. London Math. Soc.*, 59(3):373–416, 1989.

[10] Mattia Mecchia and Bruno Zimmermann. On a class of hyperbolic 3-orbifolds of small volume and small Heegaard genus associated to 2-bridge links. *Rend. Circ. Mat. Palermo (2)*, 49(1):41–60, 2000.

[11] William H. Meeks, III and Shing Tung Yau. The equivariant Dehn's lemma and loop theorem. *Comment. Math. Helv.*, 56(2):225–239, 1981.

[12] Peter Orlik. *Seifert manifolds*. Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972.

[13] Marco Reni and Bruno Zimmermann. Extending finite group actions from surfaces to handlebodies. *Proc. Amer. Math. Soc.*, 124(9):2877–2887, 1996.

[14] Dale Rolfsen. *Knots and links*. Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.

[15] Nikolai Saveliev. *Lectures on the topology of 3-manifolds*. De Gruyter Textbook. Walter de Gruyter & Co., Berlin, revised edition, 2012. An introduction to the Casson invariant.

[16] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[17] Franz Steiger. Die maximalen ordnungen periodischer topologischer abbildungen geschlossener flächen in sich. (german). *Comment. Math. Helv.*, 8:48–69, 1935.

[18] John Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[19] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997.

[20] Chao Wang, Shicheng Wang, Yimu Zhang, and Bruno Zimmermann. Extending finite group actions on surfaces over S^3. *Topology Appl.*, 160(16):2088–2103, 2013.

[21] Chao Wang, Shicheng Wang, Yimu Zhang, and Bruno Zimmermann. Embedding surfaces into S^3 with maximum symmetry. *Groups Geom. Dyn.*, 9(4):1001–1045, 2015.

[22] Bruno Zimmermann. Über abbildungsklassen von henkelkörpern. (german). *Arch. Math. (Basel)*, 33(4):379–382, 1979.

[23] Bruno Zimmermann. Genus actions of finite groups on 3-manifolds. *Michigan Math. J.*, 43(3):593–610, 1996.

[24] Bruno P. Zimmermann. On topological actions of finite groups on S^3. *Topology Appl.*, 236:59–63, 2018.